Lower Bounds for Embedding Graphs into Graphs of Smaller Characteristic

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Abstract. The subject of *graph embeddings* deals with embedding a finite point set in a given metric space by points in another target metric space in such a way that distances in the new space are at least, but not too much more, than distances in the old space. The largest new distance to old distance ratio over all pairs of points is called the *distortion* of the embedding. In this paper, we will study the distortion $dist(G, H)$ while embedding metrics supported on a given graph G into metrics supported on a graph H of lower *characteristic*, where the characteristic $\chi(H)$ of a graph H is the quantity $E - V + 1$ (E is the number of edges and V is the number of vertices in H). We will prove the following lower bounds for such embeddings which generalize and improve lower bounds given in [10].

- If $|G| = |H|$ and $\chi(G) - \chi(H) = k$, $dist(G, H) \geq g_k - 1$
- If $\chi(G) - \chi(H) = k$, $dist(G, H) \geq \frac{g_k - 4}{3}$

Further, we will also give an alternative proof for lower bounding the distortion when probabilistically embedding expander graphs into tree metrics. In addition, we also generalize this lower bound to the case when expander graphs probabilistically embed into graphs of constant characteristic.

1 Introduction

The subject of graph embeddings deals with representing a finite point set in a metric space by points in another target metric space in such a way that the distances only increase, but not by too much. Once a given metric space can be approximately embedded into a metric space that is easier to deal with, one might get approximate solutions to problems which appeared to be hard to deal with in the original metric space. For a nice introduction to to this area refer to [7], [6] and [1].

This approach has been successfully adopted in several problems. One example would be for clustering points: here, one could embed the points at hand into some metric space such as the ℓ_2-space and try geometric algorithms that are already well known. Another example is bandwidth approximation. The first non-trivial approximation algorithm for bandwidth approximation was given in [9] using a generalization of the embeddings which form the subject of this paper. Bartal in [2] gave an algorithm to probabilistically approximate arbitrary graph metrics by tree metrics within a distortion factor of $O(\log^2(n))$, and subsequently, improved it in [3] to a factor of $O(\log n \log\log n)$. This lead to the design of polylog approximation algorithms for several problems. See [5] and [8] for examples of such problems. Charikar et al. in [4] showed how to derandomize approximation algorithms designed using probabilistic approximation by trees.

We define the *characteristic* $\chi(G)$ of a graph G as the value $E - V + 1$, where E, V are the number of edges and vertices, respectively, in G. In this paper, we examine the distortion obtained when embedding metric spaces supported on graphs into metric spaces supported on graphs of lower characteristic, both deterministically and probabilistically. Our results are stated as follows:

Previous Results: Rabinovich and Raz in [10] proved the following results on embedding a graph G into a graph H of lower characteristic. Let G and H be unweighted graphs, where $|G|$ denotes the number of vertices of G, $\chi(G)$ denotes the characteristic of G, g_k denotes the length of the kth shortest cycle in G, and $dist(G,H)$ denotes the maximum distortion over all edges resulting from the best embedding of G into H.

- If $|G| = |H|$ and $\chi(G) > \chi(H)$, $dist(G,H) \geq \frac{g_1}{3} - 1$
- If $\chi(G) > \chi(H)$, $dist(G,H) \geq \frac{g_1}{4} - \frac{1}{2}$
- If $\chi(G) - 1 > \chi(H)$, $dist(G,H) \geq \frac{g_2}{4} - \frac{3}{2}$

Whether the lower bound even in case of $|G| \neq |H|$ is $\frac{g_1}{3} - 1$ is left as a open question in [10]. Also left as open is whether one can generalize the last of the statements above.

Our Results: We prove the following results.

- If $|G| = |H|$ and $\chi(G) - \chi(H) = k$, $dist(G,H) \geq g_k - 1$
- If $\chi(G) - \chi(H) = k$, $dist(G,H) \geq \frac{g_k - 4}{3}$

The first result both generalizes and strengthens the first result in [10] stated above. The second result generalizes the second and third results in [10] and is also stronger for not too small values of g_k. Our results are obtained using topological arguments which relate cycles and linear combinations of cycles in G to corresponding structures in H, and vice versa.

2 Preliminaries

In this section we give some basic definitions that will be useful throughout the paper.

Definition 1. For any set N and a function $\rho : N \times N \to \Re$, we call (N, ρ), a *metric space* if
- $\forall x, y \in N \quad \rho(x,y) \geq 0$ with equality holding iff $x = y$
- $\forall x, y \in N \quad \rho(x,y) = \rho(y,x)$
- $\forall x, y, z \in N \quad \rho(x,y) + \rho(y,z) \geq \rho(x,z)$

Definition 2. Let (N, ρ) and (M, σ) be metric spaces and $g : N \to M$ be a function, such that:

$$\forall x, y \in N \quad \rho(x, y) \leq \sigma(g(x), g(y)) \leq D\rho(x, y)$$

The infimum of all such numbers D is called the distortion of g and is denoted by $D(g)$. In this paper, we will consider only functions g which are *expansive*, i.e., $\frac{\sigma(g(x),g(y))}{\rho(x,y)} \geq 1$, for all $x, y \in N$, $x \neq y$.

Graphs and Metric Spaces. With any edge-weighted graph $G = (V, E, W)$, one can naturally associate a metric space by defining the distance between any two vertices as the length of the shortest path between them. We call such a metric space as a metric *supported* on G. Conversely, any finite metric space can be viewed as a weighted graph. Just take a complete graph and weigh each edge with the distance between the corresponding points.

Let G be a weighted graph and (N, ρ) be a metric space. The distortion of the best embedding of G into N is denoted by $dist(G, N)$. Alternately, it is the infimum of $D(g)$, over all expansive functions g from G to N. Let S be a set of metric spaces. $dist(G, S)$ is similarly defined as infimum of $dist(G, N)$ over all metric spaces $N \in S$.

We say that a metric space (N, ρ) is *dominated* by a metric space (M, σ) through a function $g : N \to M$ if $\forall x, y \in N \;\; \rho(x, y) \leq \sigma(g(x), g(y))$.

Probabilistic Embeddings. We also consider probabilistic embeddings in this paper. A graph G is said to be α*-probabilistically* approximated by a finite set of metric spaces S, if both the following conditions hold:

- There is a probability assignment to elements (M, σ) of S such that each element with non-zero associated probability dominates G through an associated function $g_{(M,\sigma)}$.
- For every pair of vertices x, y in G, the expected distance

  $$Expectation_{(M,\sigma)} \left(\sigma(g_{(M,\sigma)}(x), g_{(M,\sigma)}(y))\right)$$

  is at most α times their distance in G.

We also denote the infimum of all such αs as $pdist(G, S)$, where the infimum is taken over all probability distributions.

Going from Discrete to Continuous. Now, we will introduce a notion which allows us to work with continuous counterparts of discrete objects like graphs and edges. This notion was introduced in [10] to prove similar lower bounds. Consider any edge (from an unweighted graph) and associate with it a unit interval with the line metric (for weighted graphs, one would use intervals of length equal to the edge weight). Now, instead of unweighted graphs we work with vertices and unit intervals connecting them, in the sense that end points of the interval are identified with the respective end points of the edges. Such a continuous structure obtained from a graph G will be called $\tilde{G}$. Note that there is a metric space naturally associated with this continuous structure. The distance between any two points (where a point could be a vertex or internal to an edge) in $\tilde{G}$ is defined as the length of the shortest path connecting them. Such paths are also called *geodesic paths*. Also, note that the distance between any two vertices in $\tilde{G}$ is the same as the distance between these vertices in G.

We can also extend any 1-1 mapping, g, from the vertices of a graph G into the vertices of a graph H, to get a continuous, many-one mapping $\tilde{g}$ from $\tilde{G}$ into $\tilde{H}$, whose restriction to the vertex set of G gives us back the original mapping g. $\tilde{g}$ is called the *continuous extension* of g, and can be obtained in several ways. We obtain $\tilde{g}$ from g using a linearly defined extension, described below.

Consider any point v in $\tilde{G}$; we show how to define $\tilde{g}(v)$. If v is a vertex in G, then $\tilde{g}(v) = g(v)$. So consider the case when v is not a vertex of G. Let (u, w) be the edge in $\tilde{G}$ containing v. There could be several geodesic paths in $\tilde{H}$ that connect $g(u)$ to $g(w)$ in $\tilde{H}$. Before defining the extension, we choose any one of these geodesic paths. Intuitively, we linearly map points on the edge (u, w) in $\tilde{G}$ to this geodesic path in $\tilde{H}$. More formally, define $d_{\tilde{G}}(u, v)$ to be the distance between u and v in $\tilde{G}$, and let:

$$\alpha = \frac{d_{\tilde{G}}(u, v)}{d_{\tilde{G}}(u, w)}.$$

Define $\tilde{g}(v)$ as the unique point y on the above chosen geodesic path associated with (u, w) such that

$$\frac{d_{\tilde{H}}(g(u), y)}{d_{\tilde{H}}(g(u), g(w))} = \alpha.$$

Using the above map, [10] gave a proof for the fact that any cycle of length n would have a distortion of $\Omega(n)$ when approximated by trees.

Lemma 1 was originally proved in [10] and will be useful in our proofs.

Lemma 1. *[10] If there exists $x, y \in \tilde{G}$ such that $d_{\tilde{G}}(x, y) \geq d$ and $\tilde{g}(x) = \tilde{g}(y)$, then $dist(G, H) \geq d - 1$. Note that x, y need not be vertices of G. They are points on the continuous extension of G.*

3 Lower bounds on Distortion

In this section, we prove a range of lower bounds on approximating a graph metric supported on a given unweighted graph G by a metric supported on a graph (possibly weighted with each edge length being at least 1) H of lower characteristic. We also prove a lower bound on approximating expander graphs by a distribution on metrics supported on graphs of lower characteristic.

3.1 Intuition

Suppose we try to show that approximating a metric supported on a cycle $\mathcal{C}$ (of say 4 vertices, $ABCDA$, in that order) by a metric supported on a tree $\mathcal{T}$ (with four vertices), incurs a distortion of at least $|\mathcal{C}| - 1$. The proof would proceed as follows.

Consider any 1-1 map g[1] from the vertices of $\mathcal{C}$ to the vertices of $\mathcal{T}$, and let $\tilde{g}$ be its continuous extension. Consider traversing $\mathcal{C}$ in the order $ABCDA$, and consider the image (under $\tilde{g}$) of this trajectory. The resulting image trajectory is a closed walk on $\mathcal{T}$ which is *self-cancelling*, i.e., each edge is traversed twice, once each in opposite directions. Next consider the image of this self-cancelling closed walk under the map $\widetilde{g^{-1}}$. Note that this image is also a self-cancelling closed walk on $\mathcal{C}$. Thus, under the map $\tilde{g} \circ \widetilde{g^{-1}}$, the circle $\mathcal{C}$ maps to a self-cancelling closed walk on $\mathcal{C}$.

Note that vertices in $\mathcal{C}$ map to themselves under $\tilde{g} \circ \widetilde{g^{-1}}$. This along with the fact that the circle $\mathcal{C}$ maps to a self-cancelling closed walk on $\mathcal{C}$ under $\tilde{g} \circ \widetilde{g^{-1}}$ implies that there exists at least one cycle edge which has a distortion of $|\mathcal{C}| - 1$ under $\tilde{g} \circ \widetilde{g^{-1}}$. Since g is expansive, the distortion under $\tilde{g}$ is at least the distortion under $\tilde{g} \circ \widetilde{g^{-1}}$. It follows that some edge of $\mathcal{C}$ has distortion at least $|\mathcal{C}| - 1$ under $\tilde{g}$ and therefore under g.

3.2 Generalization

We now make the above arguments more general. Throughout the section, we use $\chi(G)$ to denote the characteristic of a graph G, which is defined as $|E_G| - |V_G| + 1$. $\chi(G)$ is the number of independent cycles in a graph in the sense that all cycles in G can be expressed in terms of linear combinations of a *cycle basis* having $\chi(G)$ cycles in it.

1. Assume that a map g is given from V_G to V_H. We will use the continuous extension $\tilde{G}$ of G and $\tilde{g}$ of g as defined in Section 2. Let $h = g^{-1}$ and consider the maps, $\tilde{G} \xrightarrow{\tilde{g}} \tilde{H} \xrightarrow{\tilde{h}} \tilde{G}$.
2. Clearly, each cycle in the cycle basis of $\tilde{G}$ maps to a cycle in $\tilde{H}$ under the map $\tilde{g}$ (actually it will map to a closed walk in $\tilde{H}$ but cancelling out edges traversed in opposite directions will leave a cycle; this cycle could also be trivial, if all edges end up cancelling out). The total number of independent non-trivial cycles that could be obtained is clearly at most $\chi(H)$.
3. Next, consider each of the non-trivial cycles obtained above and map them back to $\tilde{G}$ using the map $\tilde{h}$. Again, we obtain a new collection of cycles in $\tilde{G}$ (by cancelling out edges traversed in opposite directions). Let dim denote the number of independent non-trivial cycles obtained in $\tilde{G}$. Clearly, dim is at most $\min\{\chi(G), \chi(H)\} = \chi(H)$ (because we started out with exactly $\chi(G)$ cycles in $\tilde{G}$ and got at most $\chi(H)$ cycles in the previous step).

[1] Any expansive map has to be 1-1, as two vertices in $\mathcal{C}$ cannot get mapped to the same vertex in $\mathcal{T}$

4. Consider each cycle C in the cycle basis of $\tilde{G}$ and consider the cycle obtained by applying the map $\tilde{f} = \tilde{g} \circ \tilde{h}$ to C and cancelling out portions traversed in opposite directions. We claim that this cycle is exactly the same as that obtained by Steps 2 and 3 above for the following reason: any part of the cycle that cancels out under the map $\tilde{g}$ will cancel out under the map $\tilde{f}$ as well. It follows that the number of independent non-trivial cycles obtained by applying $\tilde{f}$ to the cycles in the above basis is dim as well.
5. We then show that if the distortion under the map $\tilde{g} \circ \tilde{h}$ is small, then $dim > \chi(H)$, which is a contradiction. It follows that the distortion under the map $\tilde{g} \circ \tilde{h}$ must be large.
6. Finally, the proof is completed by the fact that the distortion under $\tilde{g}$ is at least the distortion under $\tilde{g} \circ \tilde{h}$, since g is expansive.

All the steps above except Step 5 are ready. We concentrate on Step 5 for proving the lower bounds in the rest of the paper. All the lower bound proofs will conclude with a contradiction showing that $dim > \chi(H)$, as required by Step 5.

Special Cycle Basis. Some of the proofs require a special kind of cycle basis for G constructed as follows.

- Process cycles in G in increasing order of length.
- Include a cycle in the basis if and only if it can not be expressed as a linear combination of cycles occurring before it.

By the definition of $\chi(G)$, the size of the basis is $\chi(G)$. A key property of the above basis is that, if the ith smallest cycle is in the basis, then all cycles smaller than it are expressible as linear combinations of basis cycles smaller than the ith cycle. This property is important in the proofs.

3.3 Lower Bounds

In this section, we will prove lower bounds on approximating a graph metric by a metric supported on a graph of lower characteristic. We use $|P|$ to denote the length of the path P. Also, we do additions of paths algebraically with the same edge cancelling out if traversed in opposite directions. Note that any vertex of G maps to itself under the map $\tilde{f} = \tilde{g} \circ \tilde{h}$ where $\tilde{g}$ is expansive and $h = g^{-1}$.

Theorem 1. *If $|G| = |H|$ and $\chi(H) < \chi(G)$, then $dist(G, H) \geq g_1 - 1$, where g_1 is length of the shortest cycle in G.*

Proof: Let g denote any expansive map from the vertices of G to that of H and suppose the distortion under g is less than $g_1 - 1$. We show a contradiction as follows.

Consider any cycle basis B of G. Consider any cycle $C_i \in B$. For each edge e_i^j in C_i,

$$|e_i^j - \tilde{f}(e_i^j)| \leq |e_i^j| + |\tilde{f}(e_i^j)| \leq |e_i^j| + |\tilde{g}(e_i^j)| < 1 + (g_1 - 1).1 = g_1$$

The second inequality follows from the fact that g is expansive, and therefore distances between vertices of H can only decrease under the map $\tilde{g^{-1}}$. Next, note that a consequence of the above is that e_i^j and $-\tilde{f}(e_i^j)$ together cannot form a cycle in G as the length of any cycle must be at least g_1. Since $e_i^j - \tilde{f}(e_i^j)$ is a closed walk, $e_i^j - \tilde{f}(e_i^j) = 0$ or simply, $\tilde{f}(e_i^j) = e_i^j$.

It follows that each cycle C_i in the basis being considered maps to itself under the map $\tilde{f}$, and therefore $dim = \chi(G) > \chi(H)$, as required. □

Theorem 2. *If $|G| = |H|$ and $\chi(G) - \chi(H) = k$, then $dist(G, H) \geq g_k - 1$, where g_k is the length of the kth smallest cycle in G.*

Proof: Let g denote any expansive map from the vertices of G to that of H and suppose the distortion under g is less than $g_k - 1$. We show a contradiction as follows. Let S be the special cycle basis which we constructed in the previous subsection. Consider any cycle $C_i \in S$ such that $i \geq k$. Let,

$$\forall i, j \qquad e_i^j - \tilde{f}(e_i^j) = F_i^j$$

By arguing as in Theorem 1 we get,

$$\forall i, j \qquad |F_i^j| < g_k$$

So F_i^j must be either one of the smallest $k-1$ cycles or must completely cancel out. Then, $\sum_j \tilde{f}(e_i^j)$ cannot be written as a linear combination of cycles smaller than C_i, otherwise, since $C_i = \sum_j \tilde{f}(e_i^j) + \sum_j F_i^j$ and since $C_i \neq 0$, C_i becomes a linear combination of smaller cycles, a contradiction. Further, since $\tilde{f}(C_i) = \sum_j \tilde{f}(e_i^j)$ can be written as a linear combination of C_i and cycles smaller than C_i, the cycles $\tilde{f}(C_i)$ are all independent for $i \geq k$. This implies that $dim \geq \chi(G) - k + 1 > \chi(H)$, as required. □

Theorem 3. *If $\chi(G) - \chi(H) = k$, then $dist(G, H) \geq \frac{g_k}{3} - \frac{4}{3}$*

The main bottleneck for proving this theorem seems to be proving that each edge together with it's image under $\tilde{f}$ forms a cycle of small length. The reason why the argument in Theorem 2 does not work directly is because of the presence of extra vertices in H. The distance between the extra vertices need not contract when mapped back to G. This is the reason we get a weaker lower bound compared to the case when the number of vertices in G and H is the same.

Proof of Theorem 3: Let g denote any expansive map from the vertices of G to that of H and suppose the distortion under g is less than $\frac{g_k}{3} - \frac{4}{3}$.

To prove this theorem, we will add additional vertices to G and H. We will continue to call the resulting graphs G and H for convenience. Since the additional vertices will be internal to edges, each addition will increase the number of vertices and number of edges by 1, and therefore, there will be no change in the characteristic of the graphs in question.

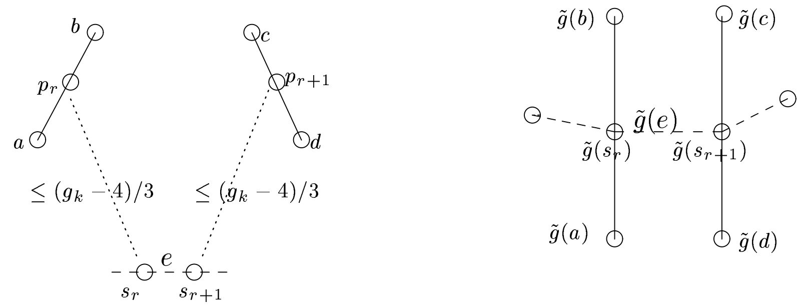

Fig. 1. G on the left and H on the right

First, we will add vertices to H such that distance between any two adjacent vertices is at most 1. This can be done by putting vertices at intervals of 1 on each edge. Second, for each vertex v in H to which no vertex in G maps under g, a point on some edge of $\tilde{G}$ that maps to v under the map $\tilde{g}$ is made a vertex in G, provided such a point exists. Note that there could be several such points in $\tilde{G}$, in which case any one is chosen. Finally, the map g from the original vertices of G to the original vertices of H is now extended to include the new vertices of G. Note that the extended map is still 1-1 (i.e., two vertices in G do not map to the same vertex in H). Therefore, $h = g^{-1}$ is well-defined. As before, we consider $\tilde{g}$, $\tilde{h}$, and $\tilde{f} = \tilde{g} \circ \tilde{h}$.

We will assume that any two points in $\tilde{G}$ mapped to the same point in $\tilde{H}$ by $\tilde{g}$ are at a distance at most $\frac{g_k}{3} - \frac{4}{3}$ in $\tilde{G}$, otherwise the proof follows by applying Lemma 1 in Section 2. We will derive a contradiction as follows.

The proof is similar to the proof of Theorem 2, with some key differences. Recall that S denotes the special basis constructed earlier and consider any cycle $C_i \in S$ such that $i \geq k$. Let e_i^j be an edge of this cycle. Mark points $s_1, \ldots, s_l$ on e_i^j that are inverse images of vertices in $\tilde{H}$ which lie on the geodesic $\tilde{g}(e_i^j)$ in $\tilde{H}$. We call the part of the edge between s_r and s_{r+1}, for $r = 1, \ldots, l-1$, an *edgelet*. Instead of working with edges, we will work with edgelets. For the rest of the proof, redefine e_i^j to be the edgelet (s_r, s_{r+1}). Let $p_r = \tilde{f}(s_r)$ and $p_{r+1} = \tilde{f}(s_{r+1})$ be vertices in $\tilde{G}$. See Figure 1.

Next, we would like to show (as in Theorem 2) that the quantity $|e_i^j - \tilde{f}(e_i^j)|$ is strictly less than g_k for each edgelet. However, this alone is not sufficient as p_r need not equal s_r and similarly p_{r+1} need not equal s_{r+1} and consequently, $e_i^j - \tilde{f}(e_i^j)$ need not form a closed walk (in contrast, recall that in earlier theorems, for each vertex v in G, $\tilde{f}(v) = v$). To get around this, we will consider instead, for each edgelet (s_r, s_{r+1}), the walk obtained by starting from s_r, taking the geodesic (in $\tilde{G}$) to p_r, and then to p_{r+1}, and then to s_{r+1}; this walk replaces $\tilde{f}(e_i^j)$ and will be called $\tilde{f}'(e_i^j)$. We will now show that $|e_i^j - \tilde{f}'(e_i^j)| < g_k$. Once this is shown, using the additional fact that the sum of $\tilde{f}'(e_i^j)$ over all edgelets in C_i is identical to the sum of $\tilde{f}(e_i^j)$ over all edgelets in C_i, the rest of the proof is identical to Theorem 2.

To show that $|e_i^j - \tilde{f}'(e_i^j)| < g_k$, it suffices to show that $|\tilde{f}'(e_i^j)| < g_k - 1$. Clearly, $|\tilde{f}'(e_i^j)| \leq d_{\tilde{G}}(s_r, p_r) + d_{\tilde{G}}(p_r, p_{r+1}) + d_{\tilde{G}}(p_{r+1}, s_{r+1})$. Note that $\tilde{g}(s_r) = \tilde{g}(p_r)$, and likewise for s_{r+1}, p_{r+1}. Then, by our assumption above, we know that $d_{\tilde{G}}(s_r, p_r) \leq \frac{g_k}{3} - \frac{4}{3}$ and $d_{\tilde{G}}(s_{r+1}, p_{r+1}) \leq \frac{g_k}{3} - \frac{4}{3}$. It follows that

$$|\tilde{f}'(e_i^j)| \leq d_{\tilde{G}}(p_r, p_{r+1}) + \frac{2g_k}{3} - \frac{8}{3}$$

Next, we bound the first term on the right hand side.

Let $p_r \in$ edge (a,b) and $p_{r+1} \in$ edge (c,d), where $(a,b), (c,d)$ are the original edges in G, i.e., these edges were present even before additional vertices were added at the beginning of this proof. Note that

$$d_{\tilde{H}}(\tilde{g}(a), \tilde{g}(b)) = d_{\tilde{H}}(\tilde{g}(a), \tilde{g}(s_r)) + d_{\tilde{H}}(\tilde{g}(s_r), \tilde{g}(b)) < \frac{g_k}{3} - \frac{4}{3}$$
$$d_{\tilde{H}}(\tilde{g}(c), \tilde{g}(d)) = d_{\tilde{H}}(\tilde{g}(c), \tilde{g}(s_{r+1})) + d_{\tilde{H}}(\tilde{g}(s_{r+1}), \tilde{g}(d)) < \frac{g_k}{3} - \frac{4}{3}$$

because of our assumption of small distortion for the original edges in G. Therefore,

$d_{\tilde{H}}(\tilde{g}(a), \tilde{g}(c)) + d_{\tilde{H}}(\tilde{g}(b), \tilde{g}(d))$
$\leq d_{\tilde{H}}(\tilde{g}(a), \tilde{g}(s_r)) + d_{\tilde{H}}(\tilde{g}(s_r), \tilde{g}(b)) + d_{\tilde{H}}(\tilde{g}(c), \tilde{g}(s_{r+1})) + d_{\tilde{H}}(\tilde{g}(s_{r+1}), \tilde{g}(d)) +$
$2d_{\tilde{H}}(\tilde{g}(s_r), \tilde{g}(s_{r+1}))$
$< \frac{2g_k}{3} - \frac{8}{3} + 2 = \frac{2g_k}{3} - \frac{2}{3}$

The second inequality above is due to the fact that the distance between consecutive vertices in $\tilde{H}$ is at most 1, by construction. Next, assuming wlog that $d_{\tilde{H}}(\tilde{g}(a), \tilde{g}(c)) \leq d_{\tilde{H}}(\tilde{g}(b), \tilde{g}(d))$ and using the fact that $\tilde{g}$ is expansive on the original edges of G, we get: $d_{\tilde{G}}(a,c) \leq d_{\tilde{H}}(\tilde{g}(a), \tilde{g}(c)) < \frac{g_k}{3} - \frac{1}{3}$. Since $d_{\tilde{G}}(p_r, p_{r+1}) \leq d_{\tilde{G}}(a,c) + 2 < \frac{g_k}{3} - \frac{1}{3} + 2$, we get that

$$|\tilde{f}'(e_i^j)| < \frac{g_k}{3} - \frac{1}{3} + 2 + \frac{2g_k}{3} - \frac{8}{3} = g_k - 1$$

and therefore $|e_i^j - \tilde{f}'(e_i^j)|$ is strictly less than g_k, as required. □

Next, we use Yao's Lemma (stated below) to prove probabilistic lower bounds on embedding expander graphs in graphs of constant characteristic.

Theorem 4. *Let us denote by $\{p\}$, a probability distribution over metric spaces from S and by $\{q\}$, a probability distribution over $(u,v) \in E_G$. Then*

$$\min_{\{p\}} \max_{(u,v)\in E_G} \sum_{H\in S} p_H \frac{d_H(u,v)}{d_G(u,v)} \geq \max_{\{q\}} \min_{H\in S} \sum_{(u,v)\in E_G} q_{uv} \frac{d_H(u,v)}{d_G(u,v)} \qquad (1)$$

Theorem 5. *Let G be a graph of n vertices, m edges and girth g_1, and let S be a family of graphs with characteristic at most $\chi(G) - k(n)$. Consider embeddings of G into each of the graphs in S. For any probability distribution over S, there must be an edge in G whose expected distortion under the above embeddings is at least $\frac{k(n)(g_1-7)}{3E_G} + 1$.*

Proof: We use Theorem 4 with uniform distribution over the edges of G. Fix any $H \in S$ and consider the map g embedding G in H. We show that there must be at least $k(n)$ edges having a distortion at least $\frac{g_1-4}{3}$ under g. This being the case, the average distortion over all edges is, as required, at least:

$$\frac{1}{|E_G|}[k(n)\frac{g_1-4}{3} + (E_G - k(n))1] \geq \frac{k(n)(g_1-7)}{3E_G} + 1$$

For a contradiction, suppose there are $l < k(n)$ edges having distortion at least $\frac{g_1-4}{3}$. Remove these edges and consider the remaining graph G'. The characteristic of this graph is $\chi(G) - l > \chi(G) - k(n) \geq \chi(H)$ and its girth is at least g_1. Every edge in G', and consequently, every pair of vertices in G' has distortion less than $\frac{g_1-4}{3}$ in the embedding of G' in H. But by Theorem 3, since $\chi(G') > \chi(H)$, there must be a pair of vertices in G' with distortion at least $\frac{g_1-4}{3}$, a contradiction. □

Corollary 31 *An Expander graph with degree c and girth $g_1(c)\log n$ cannot be probabilistically approximated strictly within a factor of $\frac{k(n)(g_1(c)\log n-7)}{1.5cn} + 1$ by metrics over graphs having characteristic lower by at least $k(n)$. In particular, it cannot be probabilistically approximated within an $\Omega(\log n)$ factor, by graphs of constant characteristic.*

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