Short-cuts on Star, Source and Planar Unfoldings

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Abstract. When studying a 3D polyhedron, it is often easier to cut it open and flatten in on the plane. There are several ways to perform this unfolding. Standard unfoldings which have been used in literature include *Star* Unfoldings, *Source* Unfoldings, and *Planar* Unfoldings, each differing only in the cuts that are made. Note that every unfolding has the property that a straight line between two points on this unfolding need not be contained completely within the body of this unfolding. This could potentially lead to situations where the above straight line is shorter than the shortest path between the corresponding end points on the polyhedron. We call such straight lines *short-cuts*. The presence of short-cuts is an obstacle to the use of unfoldings for designing algorithms which compute shortest paths on polyhedra. We study various properties of Star, Source and Planar Unfoldings which could play a role in circumventing this obstacle and facilitating the use of these unfoldings for shortest path algorithms.
We begin by showing that Star and Source Unfoldings do not have short-cuts. We also describe a new structure called the *Extended Source* Unfolding which exhibits a similar property. In contrast, it is known that Planar unfoldings can indeed have short-cuts. Using our results on Star, Source and Extended Source Unfoldings above and using an additional structure called the *Compacted Source* Unfolding, we provide a necessary condition for a pair of points on a Planar Unfolding to form a short-cut. We believe that this condition could be useful in enumerating all Shortest Path Edge Sequences on a polyhedron in an output-sensitive way, using the Planar Unfolding.

1 Introduction

Star Unfoldings have proved useful tools in arguing about shortest paths on the surface of a 3D polytope $\mathcal{P}$ (see Fig. 1(a)). A Star Unfolding is obtained by cutting a 3D polytope along shortest paths emanating from a reference point x and going to each of the vertices of $\mathcal{P}$; the resulting structure is then flattened on the plane. We denote the Star Unfolding with respect to the reference point x by $star(x)$. The idea of such an unfolding is due to Aleksandrov [1], who uses it to show that $\mathcal{P}$ can be triangulated. Aronov and O'Rourke [10] showed that this unfolding does not self-overlap (in fact they show a stronger property, which we shall describe shortly). Subsequently, Agarwal, Aronov, O'Rourke and Schevon

[12] showed how the Star Unfolding can be used for enumerating Shortest Path Edge Sequences on a convex polyhedron and for finding the geodesic diameter of a convex polyhedron. Chen and Han [11], independently, used Star Unfoldings for computing shortest path information from a single fixed point on the surface of a polytope.

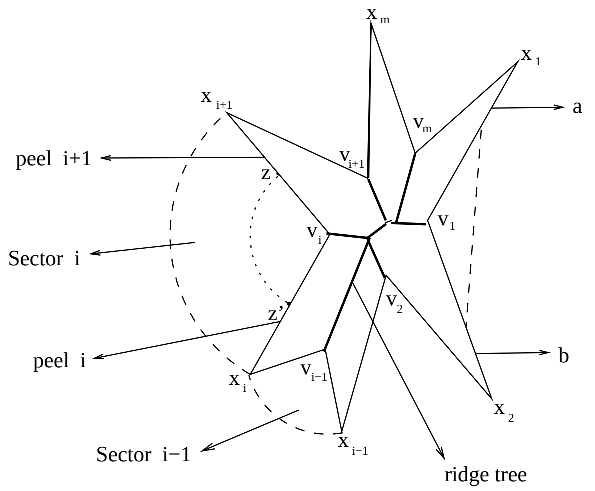

Fig. 1. Star Unfolding

The *Source Unfolding* of a 3D polytope $\mathcal{P}$ is obtained by transforming the Star Unfolding as follows. The Star Unfolding $star(x)$ (with respect to the reference point x) has a *ridge tree* [7] comprising all points which have two or more distinct shortest paths from the reference point x. The leaves of this ridge tree are the vertices of $\mathcal{P}$ (x aside) and internal ridge nodes are those points on $\mathcal{P}$ which have 3 or more shortest paths to x. Cutting along the ridge tree breaks the Star Unfolding into several convex polygons, called *peels*. The Source Unfolding is obtained by rearranging these peels around x. We denote the Source Unfolding with respect to the reference point x by $source(x)$. In Fig. 2(a) and (b), we have shown the Source Unfolding of $\mathcal{P}$ with respect to x for the cases when x is not a vertex on $\mathcal{P}$ and when x is a vertex of $\mathcal{P}$, respectively. This unfolding was defined in [3]. Note that in case 1, x is completely surrounded by the body of the unfolding while in the other case, there is an empty angular region incident on x due to the positive curvature of the polyhedron at x.

We define *Planar Unfoldings* as follows. Consider any sequence of distinct faces $f_1 \dots f_k$ on the surface of $\mathcal{P}$ such that f_i and f_{i+1} share an edge e_i, for all i, $1 \leq i \leq k-1$. Consider cutting out $f_1 \dots f_k$ from $\mathcal{P}$ by cutting along each edge that is on the boundary of some f_i, with the exception of the edges $e_1 \dots e_{k-1}$. Note that by virtue of $f_1 \dots f_k$ being distinct, cuts are made along

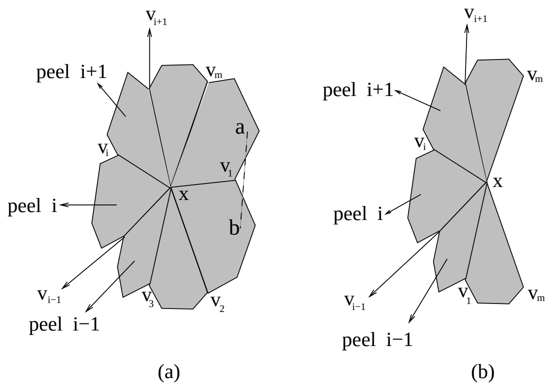

Fig. 2. Source Unfolding

all but exactly two of the boundary edges for each f_i, $2 \leq i \leq k-1$, (the exceptions being e_{i-1}, e_i for f_i). Further, for f_1 and f_k, cuts are made along all but exactly one of the boundary edges (the exceptions being e_2, e_{k-1}). With these properties, it is easy to see that the resulting structure can be flattened out on a plane (because, after cutting out, f_i and f_{i+1} are attached to each other by exactly one edge, around which we can rotate to get f_i and f_{i+1} on one plane). We call this structure the Planar Unfolding with respect to the edge sequence $e_1 \ldots e_{k-1}$.

Our Results. Our contribution is four-fold, all using only elementary geometry.

1. Aronov and O'Rourke [10] showed that the Star Unfolding does not self-overlap. Further, they showed a stronger property: the circular sectors between adjacent peels (see Fig.1(a)) are also disjoint. As a corollary of this property, they show that there are no short-cuts in the Star Unfolding with one endpoint being the reference point x. In other words, a straight line between any instance of x in the Star Unfolding and any other point a in the Star Unfolding is no shorter than the shortest path between x and a on $\mathcal{P}$. We extend this result to all pairs of points on the Star Unfolding. So, for any pair of points a and b on the Star Unfolding, we show that the straight line joining these two points is no shorter than the shortest path between a and b on $\mathcal{P}$.

2. Using the above result that the Star Unfolding has no short-cuts, we show that the Source Unfolding also does not have short-cuts. The proof uses a stepwise transformation of the Star Unfolding into the Source Unfolding.

3. Next, using the above result for Source Unfoldings, we show that the extension (with respect to any point y) of the Source Unfolding does not have short-cuts, i.e., for any point a on the extended line xy in the Extended Source Unfolding and any other point b on the Source Unfolding, the straight line joining a and

b is no shorter than the shortest path between a and b on $\mathcal{P}$. This notion of *Extended Source* Unfoldings will be defined later.

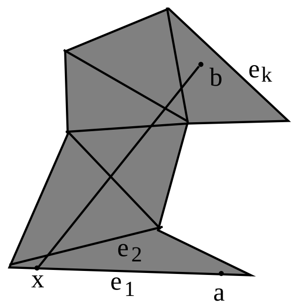

Fig. 3. Planar Unfolding

4. It is known [5] (and we give an example to this effect) that, unlike Star and Source Unfoldings, Planar Unfoldings could have short-cuts, i.e., the distance between a pair of points x and y on a Planar Unfolding can be shorter than the distance between the same pair of points on the convex polyhedron. However, the above result on Extended Source Unfoldings gives the following necessary condition for a pair of points to form a short-cut on a Planar Unfolding.

A Necessary Condition for a Short-Cut on Planar Unfoldings. Consider three points x, a, b on some Planar Unfolding of $\mathcal{P}$ (see Fig.3) such that xa and xb are images of geodesics completely within this unfolding. For ab to be a short-cut in this Planar Unfolding, neither xa nor xb must be a shortest path on $\mathcal{P}$.

In the above process, we also define a new structure called the *Compacted Source Unfolding* by refolding the Source Unfolding in a certain way, and then use this structure to show properties of shortest paths which are then used to prove the above claims on Star, Source, Extended Source and Planar Unfoldings.

Applications and Further Work. Our motivation for studying short-cuts was to obtain an output-sensitive algorithm for enumerating SPES's (i.e., shortest path edge sequences, sequences of edges which are crossed by at least one shortest path) on the polyhedron P. Unlike previous work [12, 7] which uses the Star Unfolding as its main tool, our approach was to use the Planar Unfolding, where a particular edge sequence is considered if and only if all its prefixes are SPES's; this would lead naturally to output-sensitivity, as only SPES's are considered for further extension using a Dijkstra like scheme. The first hurdle encountered in this process is that of short-Cuts. While we haven't yet obtained a complete output-sensitive algorithm for enumerating SPES's, we believe that the properties of short-cuts listed above could be useful in obtaining such an algorithm. In addition, in a companion paper [13, 14], we make heavy use of these properties to prove some more interesting properties of Planar Unfoldings, described below.

Consider a Planar Unfolding of the faces $f_1 \ldots f_k$ forming the edge sequence $e_0 \ldots e_k$, where e_i is common to f_i and f_{i+1}, for all i, $1 \leq i \leq k-1$. Further, suppose $e_0 \ldots e_{k-1}$ is an SPES crossed by shortest path a, b and $e_1 \ldots e_k$ is an SPES crossed by shortest path a', b'. Then ALL short-cuts between e_0 and e_k lie completely on one particular side of the infinite straight line passing through a and b (wlog, keeping this line vertical with a below b, short-cuts are either all to the left of this line or they are all to the right of this line). Further, if xy is a short-cut to the right of ab with $x \in e_0$ and $y \in e_k$, then either $source(a)$ has all shortest paths ay in the clockwise region between the shortest paths ab' and ax, or $source(b')$ has all shortest paths $b'x$ in the clockwise region between the shortest paths $b'y$ and $b'a'$,

Road map. Section 2 lays out some preliminaries. Section 3 defines Compacted Source Unfoldings and proves some properties needed in later sections. Sections 4, 5 and 6, prove that there are no short-cuts on Star, Source and Extended Source Unfoldings, respectively. Section 11 provides an example for the presence of short-cuts in Planar Unfoldings. Section 7 shows our necessary condition for a pair of points on a Planar Unfolding to form a short-cut. For want of space, Section 11,

2 Definitions and Preliminaries

Let $\mathcal{P}$ denote a 3D polyhedron. We use the term *vertex* of $\mathcal{P}$ to denote one of the corners of $\mathcal{P}$. For any two points a, b on $\mathcal{P}$, let $SP(a, b)$ denote the shortest path between a and b on $\mathcal{P}$. For any two points a, b on the plane, let ab denote both the straight line segment connecting a, b and the length of this straight line segment as well (for convenience). Let $C(a, ab)$ denote the circle with center a and radius ab.

Definitions. For any two points a, b on the Star Unfolding, let $star(a, b)$ be the length of the straight line ab. $source(a, b)$ and $planar(a, b)$ are defined analogously.

Aronov and O'Rourke [10] showed that the Star Unfolding can be embedded in the plane without overlap. The resulting structure is a simple polygon (see Fig.1(a)) whose vertices in cyclic order are denoted by $x_1, v_1, x_2, v_2, \ldots, x_m, v_m$, where $x_1 \ldots x_m$ are images of the source point x and $v_1 \ldots v_m$ are the vertices of $\mathcal{P}$ (excluding x, if it is a vertex). Note that for each i, $1 \leq i \leq m$, $x_i v_i = v_i x_{i+1}$ (here, and in what follows, $i+1$ is taken modulo m, implicitly). Aronov and O'Rourke actually showed the following stronger fact in Lemma 1, for which we need the following definition. Let $Sector(i)$ denote the sector of the circle $C(v_i, x_i v_i)$ which lies between the lines $x_i v_i$ and $x_{i+1} v_i$ (see Fig.1).

Lemma 1. *[10] The various sectors $Sector(1) \ldots Sector(m)$ are mutually pairwise disjoint; further these sectors lie completely outside the Star Unfolding. It follows that the Star Unfolding does not self-overlap and is a simple polygon.*

Lemma 1 yields the following lemma on peels (recall, peels were defined in the introduction).

Lemma 2. *For any point y on $star(x)$, the Star Unfolding with respect to x, there exists an x_i, $1 \leq i \leq m$, such that the straight line x_iy is contained completely within the peel containing x_i; further, $star(x_iy) = SP(xy)$.*

Fact 1 *Consider two points a, b on either a Star, Source or Planar Unfolding such that the straight line joining them lies completely inside the unfolding (see Fig.1(b) for an example). Then, this straight line is an image of a geodesic on the surface of $\mathcal{P}$ and therefore, has length at least $SP(a, b)$.*

Triangulating $star(x)$ and Triangle Images. Note that if we triangulate the peels in $star(x)$ using triangles whose one endpoint is always some x_i, then each such triangle has a congruent *image triangle*; this is illustrated in the picture below where t'_j is the image of t_j and t'_i is the image of t_i. This notion of image of a triangle will be used in defining Compacted Source Unfoldings in the next section. Note that the image pairing of triangles has a nesting property, namely that the various angular stretches between triangles and their respective images are either mutually disjoint or one stretch contains the other.

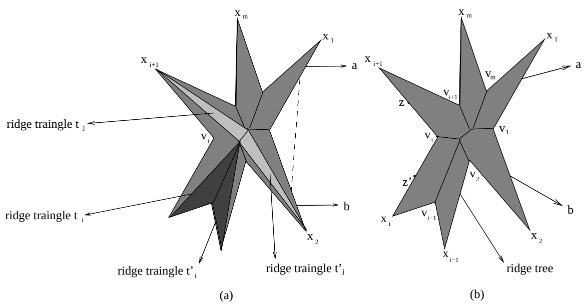

Fig. 4. Star Unfolding: Triangle Images

3 The Compacted Source Unfolding

In this section we define a new structure called the *Compacted Source* Unfolding which is obtained by folding the Source Unfolding as follows. Consider $source(a)$, the Source Unfolding with respect to a and suppose one is given any two points u and v on this unfolding such that $SP(au)$ and $SP(av)$ are maximal shortest paths in $source(a)$, Consider the region of $source(a_{uv})$ which lies clockwise from $SP(au)$ to $SP(av)$ and consider the triangles into which peels in this region are triangulated. Some of these triangles will have their images within this region as well, while others will not. Fold $source(a_{uv})$ in such a way that each triangle which has its image in this region gets pasted to its image, back to back (this is possible because of the nesting property described above). The resulting structure will have a *backbone*, comprising those triangles which do not have images in this region, and *ribs* comprising triangles which indeed have images in this region, with the images pasted back to back; this is the *Compacted Source* unfolding with respect to a and u, v.

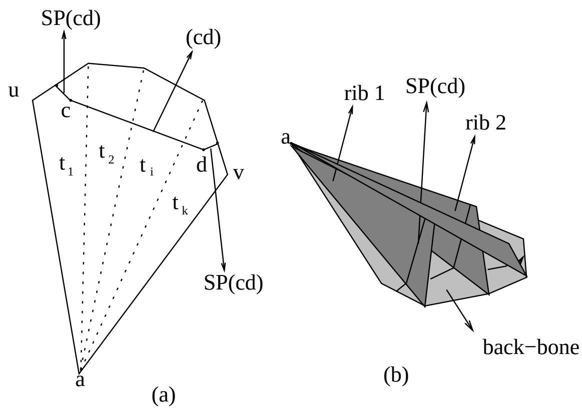

Fig. 5. The Compacted Source Unfolding: Backbones and Ribs

We can show the following properties on this Compacted Source Unfolding. Theorem 1 and its corollary below follow from the fact that a path which goes through the ribs can be shortened to one that stays on the backbone but has the same source and destination. Theorem 2 is the key non-trivial claim that we prove. Proofs appear in the Appendix.

Lemma 3. *The back-bone of* $source(a_{(uv)})$ *is a convex polygon.*

Theorem 1. *Every pair of points on a peel is joined by a unique shortest path on the convex polyhedron thats lies entirely on that peel.*

Corollary 1. *The intersection of a shortest path with a peel is connected.*

Theorem 2. *Consider a portion* $source(a_{(uv)})$ *of* $source(a)$*, where* u *and* v *are any pair of ridge points such that the angle* $\angle vau < \pi$ *(Fig.14(a)). Let* (uv) *denote the line segment that joins* u *and* v *on the plane that contains* $source(a)$ *and suppose that* $SP(av) \geq SP(au)$*. If* (uv) *lies completely outside the Source Unfolding (except for its endpoints), then there exists another shortest path* $SP(au)$ *in* $source(a_{(uv)})$*.*

4 Star Unfoldings

We prove the following theorem in this section.

Theorem 3. *Consider any two points* a, b *on the Star Unfolding with respect to point* x *on* $\mathcal{P}$*. Then* $star(ab) \geq SP(ab)$*.*

The proof outline is as follows: By Fact 1, this is true if the line segment ab is completely inside $star(x)$. It now suffices to consider the case when ab is completely outside $star(x)$, for the only remaining case can be handled by splitting ab into internal and external segments and arguing on each segment. For the case when ab is completely outside as shown in Fig.6, using Lemma 1, one can show that the path a to x_i and then x_j to b on the polyhedron $\mathcal{P}$ is shorter than ab, as required. The full proof appears in the appendix.

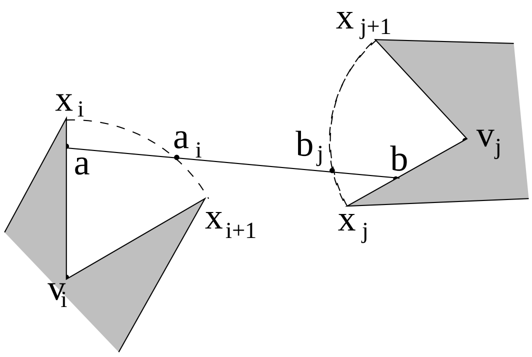

Fig. 6. a, b completely outside

5 Source Unfoldings

Theorem 4. *Consider any two points* a, b *on the Source Unfolding with respect to point* x *on* $\mathcal{P}$*. Then* $source(ab) \geq SP(ab)$*.*

Note that as in the case of $star(x)$, we need only consider the case when the line segment ab is completely outside $source(x)$. The proof proceeds by induction on the number of peels (call this the *peel distance*) in $source(x_{ab})$. If this is one or two, then the argument is easily made by rotating peels from the source configuration to the star configuration while showing that distance ab only decreases in this process; this along with Theorem 3 finishes the proof. If the peel distance is more than two, then Theorem 2 gives us a tool to perform induction.

Without loss of generality, we assume that the angle at x in $source(x_{ab})$ is $< \pi$ and that $xb \leq xa$ on $source(x)$. Then, by Theorem 2, there exists another copy b' of b in $source(x_{ab})$. We then argue inductively on ab' instead of ab, as the former has a provably smaller peel distance. The full proof appears in the Appendix.

6 Extended Source Unfoldings

We define the *Extended Source Unfolding* of $source(x)$ constructively as the structure obtained from the Source Unfolding by taking a shortest path xy on $source(x)$ (which also corresponds to a shortest path on the polyhedron $\mathcal{P}$) and then extending it as a geodesic for a finite stretch or until it reaches a vertex of $\mathcal{P}$. We prove the following theorem in this section.

Theorem 5. *Consider the Source Unfolding $source(x)$ with respect to x. Let xb and xy be any pair of shortest paths. Let a be any point that lies on a geodesic that is a finite extension of the shortest path xy. Then $source(ab) \geq SP(ab)$.*

Proof. Let us assume that xa in the Extended Source Unfolding is along the x-axis (in a Cartesian coordinate system). Then we have x, y and a in that order from left to right along x-axis as shown in Fig.7. We have the following two cases to consider.

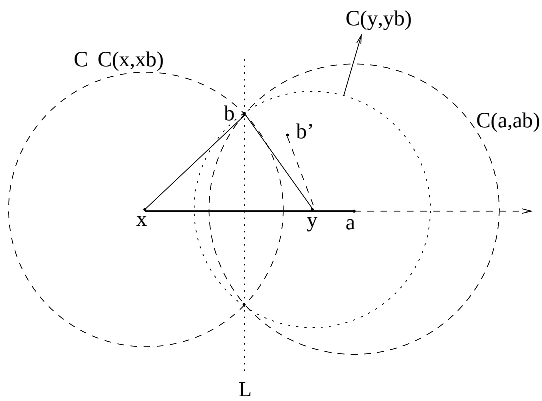

Fig. 7. $source(ab) \geq SP(ab)$ in extended source unfolding: Case 1

Case 1: The portion ya of xa is a shortest path in $source(y)$: Let $source(x)$, $source(y)$ and $source(a)$ denote the Source Unfoldings of the polyhedron with respect to x, y and a respectively. Let us superimpose $source(y)$ and $source(a)$ on the line segment xa in the Extended Source Unfolding of $source(x)$ such that the following conditions are satisfied.

- The portions xy (a shortest path) and ya (a shortest path) of xa in the Extended Source Unfolding coincide with the shortest paths xy and ya in $source(y)$. This is always possible for the following reason. Since xa is an extension of the shortest path xy in the Extended Source Unfolding, y cannot be a polyhedral vertex (according to the definition of the Extended Source Unfolding). Hence, a small neighborhood of y in the Extended Source Unfolding is identical to a small neighborhood of y in $source(y)$. Thus, the angle between the shortest paths ya and yx in each of the Extended Source Unfolding and $source(y)$ is equal to π.
- The portion ya (a shortest path) of xa coincides with the shortest path ya in $source(a)$.

Let b' denote the position of b in $source(y)$. That is, a shortest path from y to b in $source(y)$ is denoted by yb'. Consider the three discs $C(x, xb), C(y, yb)$ and $C(a, ab)$ (we define the disc $C(x, xb)$ as the set of points p such that $xp \leq xb$, other discs are defined similarly). Since xy and xb are shortest paths, according to Theorem 4, $yb \geq SP(yb)$. Thus, the shortest path from y to b, that is yb', in $source(y)$ is shorter than yb. Thus, b' lies on the disc $C(y, yb)$ (Fig.7). Since yx and yb' are shortest paths in $source(y)$, $xb' \geq SP(xb') = SP(xb)$. Hence, b' does not lie inside the disc $C(x, xb)$ (because the radius of this disc is a shortest path). Thus we have b' on the portion of the disc $C(y, yb)$ that is not inside $C(x, xb)$. Hence, b' cannot lie to the left of the line segment L through b that is normal to the x-axis. But this portion of the disc $C(y, yb)$ (that contains b') is on the disc $C(a, ab)$ because a is to the right of y. Thus, we have b' on or inside $C(a, ab)$, that is, $ab \geq ab'$.

Consider the two shortest paths ya and yb' in $source(y)$. Since, we have assumed that the portion ya of xa is a shortest path, according to Theorem 4, $ab' \geq SP(ab') = SP(ab)$. Thus, we have $ab \geq ab' \geq SP(ab)$, and the theorem holds for this case.

Case 2: ya is not a shortest path in $source(y)$: Let us partition xa into a sequence of shortest paths (Fig.8) $a_0a_1, a_1a_2, a_2a_3, ..., a_{m-1}a_m$ (where $a_0 = x$, $a_1 = y$ and $a_m = a$) such that, each partition a_ia_{i+1} is the portion of ya that lies on a single peel. Each of these partitions satisfies the following two properties.

- Each partition is a shortest path (we will be showing in Theorem 1 that any geodesic that lies inside a peel is a shortest path).
- We will show that the number of peels crossed by any finite geodesic is countable. Hence, the number of partitions in xa is also countable.

The number of peels crossed by any finite geodesic xa is countable for the following reason. Since xa is a geodesic, there is no polyhedral vertex in the interior of xa. Thus, the intersection of a peel with xa is either null or of non-zero length. Hence, we can order the peels that intersect xa in the direction from x to a. Thus, the set of peels that have a non-empty intersection with xa is countable.

Let b_i denote the image of b in $source(a_i)$ for each $i, 0 \leq m$. Consider the three points a_i, a_{i+1}, a_{i+2} for any $i, 0 \leq i \leq m-2$. We have already mentioned that

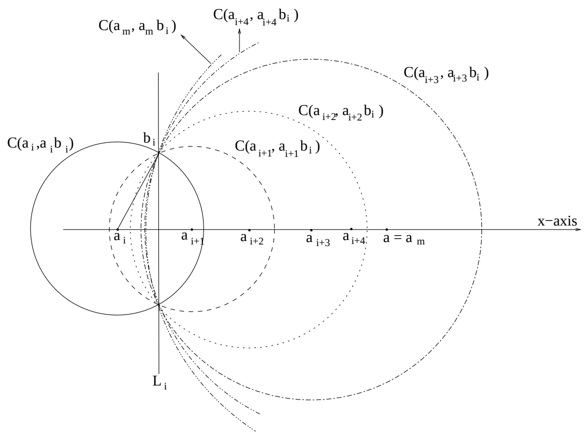

Fig. 8. $source(ab) \geq SP(ab)$ in extended source unfolding: Case 2

b_i, b_{i+1} and b_{i+2} denotes the images of the point b in the $source(a_i)$ $source(a_{i+1})$ and $source(a_{i+2})$, respectively. We have the following analogy to Case 1. We can view the four points a_i, a_{i+1}, a_{i+2} and b_i as x, y, a and b in Case 1, respectively. An argument on lines similar to that of Case 1 shows the following (see Fig.8).

Since a_ib_i and a_ia_{i+1} are shortest paths in $source(a_i)$, $a_{i+1}b_i \geq SP(a_{i+1}b_i) = SP(a_{i+1}b_{i+1}) = SP(a_{i+1}b)$. Thus, b_{i+1} lies on the disc $C(a_{i+1}, a_{i+1}b_i)$.
Since $a_{i+1}a_i$ and $a_{i+1}b_{i+1}$ are shortest paths in $source(a_{i+1})$, $a_ib_{i+1} \geq a_ib_i = SP(a_ib_i)$. Hence, b_{i+1} does not lie inside the disc $C(a_i, a_ib_i)$.
Thus, b_{i+1} lies on the disc $C(a_{i+1}, a_{i+1}b_i)$ but not inside the disc $C(a_i, a_ib_i)$. That is, b_{i+1} lies on the portion of the disc $C(a_{i+1}, a_{i+1}b_i)$ that is on or to the right of L_i where L_i is the normal from b_i to the x-axis.
Since, for each $j \geq i+2$, the point a_j lies to the right of a_{i+1}, the above mentioned portion of the disc $C(a_{i+1}, a_{i+1}b_i)$ lies inside $C(a_j, a_jb_i)$. Thus we have $a_jb_i \geq a_jb_{i+1}$ for each $j \geq i+2$. In particular we have $a_mb_i \geq a_mb_{i+1}$.
Since the above claims are true for each $i, 0 \leq i \leq m-2$, we get

$$a_mb_0 \geq a_mb_1 \geq a_mb_2 \geq a_mb_3 \geq \ldots \geq a_mb_{m-1}.$$

Since $a_{m-1}a_m$ and $a_{m-1}b_{m-1}$ are shortest paths in the Source Unfolding $source(a_{m-1})$, we have $a_mb_{m-1} \geq SP(a_mb)$. Thus we have $source(ab) = a_mb_0 \geq SP(a_mb) = SP(ab)$ (recall $a_m = a$), and we have proved the theorem for case 2. $\Box$.

7 A Necessary Condition for Short-cuts in a Planar Unfolding

Our result on Extended Source Unfoldings above provides a necessary condition for ab to be a short-cut on a Planar Unfolding, i.e., for $planar(ab) \geq SP(ab)$.

Theorem 6. *Consider three points x, a, b on some Planar Unfolding such that xa and xb are images of geodesics completely within this unfolding, and further, the xa geodesic is actually a shortest path on $\mathcal{P}$. Then $planar(ab) \geq SP(ab)$, and therefore ab cannot be a short-cut.*

Proof. Since the neighborhoods of x in the Source and the Planar Unfoldings are identical, the angle between xa and xb in these two unfoldings are the same. Hence, we have $planar(ab) = source(ab)$. However, according Theorem 5, we have $source(ab) \geq SP(ab)$. Hence, we have $planar(ab) = source(ab) \geq SP(ab)$. □

In particular note that if a lies on an edge of $\mathcal{P}$ and we take x to be on the same edge, if xa is a geodesic then ab cannot be a short-cut. Since geodesics are locally verifiable in contrast to shortest paths, this is a non-trivial condition. A simple further corollary is that any planar unfolding between edges e_0 and e_k with a geodesic from e_0 to e_k which lies completely inside this planar unfolding cannot have shortcuts which cross this geodesic.

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Appendix

8 Compacted Source Unfoldings and Proofs of Properties

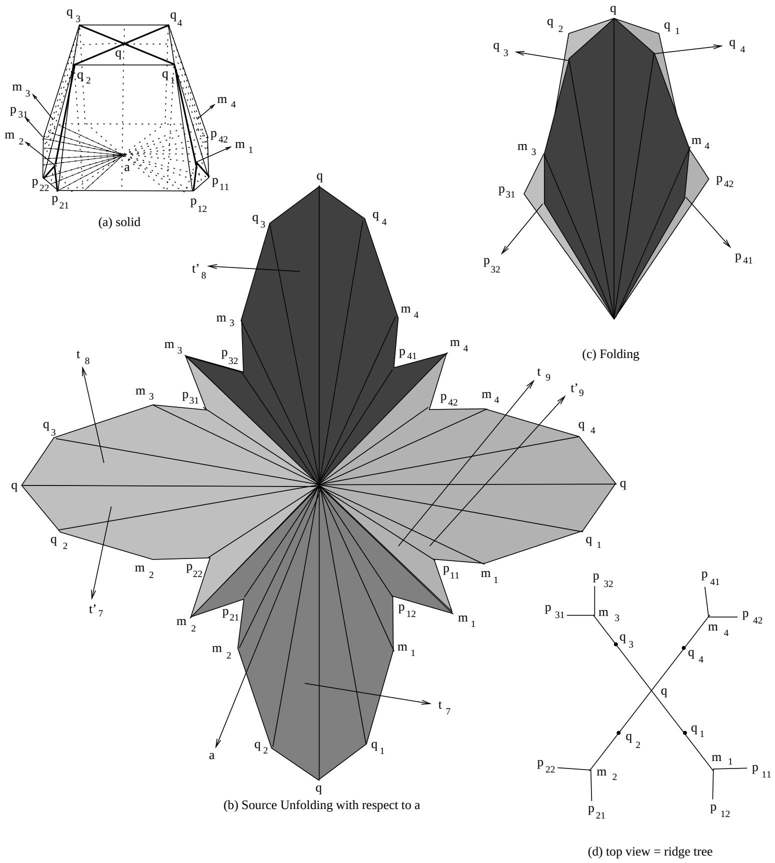


Fig. 9. Compacted Source Unfoldings

Let $source(a)$ denote the Source Unfolding of a convex polyhedron $\mathcal{P}$ with respect to a. Recall from the definition of the Source Unfolding that each point on the boundary of the Source Unfolding (except possibly the polyhedral vertices) has more than one shortest path from a on the polyhedron. Such a point is called a *ridge point* with respect to the point a (see [7]). The set of all ridge points with respect to the point a on the convex polyhedron constitutes a tree on the polyhedron called the *ridge tree* with respect to a or equivalently, the *ridge tree* with respect to $source(a)$. The ridge tree has *ridge edges* and *ridge* *nodes*. The ridge edges are geodesics on the polyhedron. Thus, each point on the interior of a ridge edge in the ridge tree is a point in the interior of a face or an edge in the polyhedron as well. Further, each point on the interior of a ridge edge has exactly two shortest paths from a. Each polyhedral vertex is a ridge node in the ridge tree. However, not all ridge nodes are polyhedral vertices. If a ridge node is not a polyhedral vertex, then it is a point on a face or an edge of the polyhedron where three or more shortest paths from a will meet. See Fig.9(a) and (b) for an example. The ridge tree is shown in bold in this figure. The polyhedral vertices $q_i, 1 \leq i \leq 4$, are internal nodes in the ridge tree whereas the polyhedral vertices $p_{ij}, 1 \leq i \leq 4$ and $1 \leq j \leq 2$ are the leaves. The ridge nodes $m_i, 1 \leq i \leq 4$, are internal points on the faces of the polyhedron each having three shortest paths from a. The ridge node q lies on a face of the polyhedron with four shortest paths from a.

Let us assume that the $source(a)$ is triangulated so that each triangle has a ridge edge as its base and a as its vertex. We call these triangles as *ridge triangles*. Note that each ridge edge is shared by exactly two ridge triangles that are similar. We say that these two similar triangles are *images* of each other. Suppose that we paste (to be specific, superimpose) each ridge triangle in the $source(a)$ to its image so that the corresponding vertices, edges and internal points coincide. We call the resulting structure as the *Compacted Source* Unfolding. Intuitively, the Source Unfolding folds like an umbrella to form its compact version. See Fig.9 for an example. We have shown the ridge tree with respect to point a in bold lines in Fig.9(a) and the corresponding Source unfolding $source(a)$ in Fig.9(b). The ridge triangles that share the ridge edges are pasted together in Fig.9(c) to get the Compacted Source. The top view of this is shown in Fig.9(d); this view is identical to the ridge tree itself.

Let ρ be the ridge tree that corresponds to $source(a)$. Choose any pair of maximal shortest paths $SP(au)$ and $SP(av)$ in $source(a)$. As we defined earlier, the $source(a_{(uv)})$ is the portion of $source(a)$ clockwise from $SP(au)$ to $SP(av)$. Since $SP(au)$ and $SP(av)$ are maximal shortest paths in $source(a)$, u and v lie on the ridge tree ρ. Let ρ_{uv} denote the path on the ridge tree from u to v. Let τ_{uv} denote the sequence of ridge triangles in the $source(a_{(uv)})$. Among the ridge triangles in τ_{uv}, some of the pairs can be mutually the images of each other. Let $\tau'_{uv} = t_1 \dots t_k$ denote the subsequence of τ_{uv} that is obtained after removing all pair of triangles in τ_{uv} that are mutually the images of each other. We call τ'_{uv} as the *back-bone* of $source(a_{(uv)})$.

See Fig.10. We have the same polyhedron as in Fig.9. We have chosen $q_4 = u$ and $p_{12} = v$. The $source(a_{(uv)})$ is the clockwise portion from aq_4 to ap_{12}. The back-bone will have the triangles $\triangle aq_4q$, $\triangle aqq_1$, $\triangle aq_1m_1$, $\triangle am_1p_{11}$, $\triangle ap_{11}m_1$ and $\triangle am_1p_{12}$. We have labelled these triangles t_1 through t_6 respectively. The back-bone is shown by a curve in bold in Fig.10(c). The back-bone consists of the triangles t_1, t_2, t_3, t_6.

Consider the back-bone formed by the sequence of triangles $t_1 \dots t_k$. For those pairs of triangles t_{i-1} and t_i that have the same shortest path (on the polyhedron) an_i as their boundary, these two triangles are adjacent in a nat-

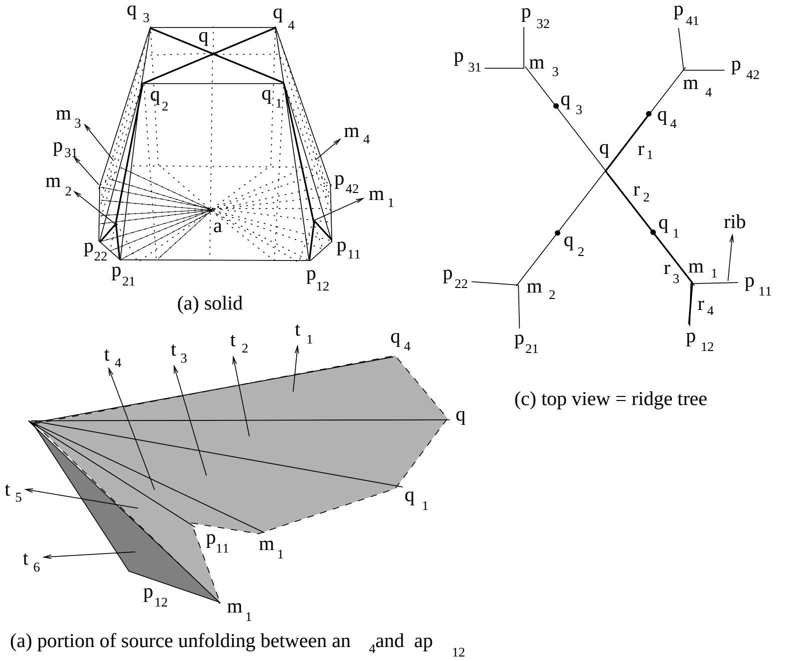


Fig. 10. Back-bone and ribs

ural way in the back-bone because they are neighbors in the polyhedron also. Whereas, those pairs of triangles t_{i-1} and t_i that have different shortest paths an_i as their boundary, a portion of the Compact Source will also be attached to their *now* common boundary an_i. This portion consists of ridge triangles that are interposed between t_{i-1} and t_i in τ_{uv} (or equivalently, $source(a_{(uv)})$). We call this portion of the Compact Source that also belongs to $source(a_{(uv)})$ and that is not part of the back-bone, but attached at an_i to the backbone, as a *rib* with respect to $source(a_{(uv)})$.

Consider the four triangles t_3, t_4, t_5 and t_6 that share the common edge am_1 in the Compact Source (see Fig.10). The triangles t_3 and t_6 are adjacent in the back-bone of $source(a_{(uv)})$, but their common boundary am_1 in the back-bone is not the same shortest path on the polyhedron. The two triangles t_4 and t_5 being mutually the images of each other, get pasted back to back constitute the rib attached to the back-bone along am_1 between t_3 and t_6.

Consider the sequence $\tau'_{uv} = t_1 \dots t_k$. Each ridge triangle has a ridge edge r_i in its boundary. Also note that r_i and r_{i+1} share the ridge vertex n_i for each triangle t_i. Let $n_0 = u, n_k = v$. Thus we have the sequence of ridge edges $r_1 \dots r_k$ corresponding to $\tau'_{uv} = t_1 \dots t_k$. The following lemma proves the relation between $r_1 \dots r_k$ and ρ_{uv}.

Lemma 4. $\rho_{uv} = r_1 \dots r_k$.

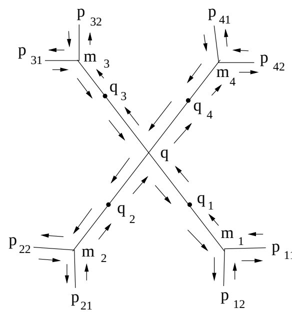


Fig. 11. depth-first traversal

Proof. Without loss of generality, we shall assume that the ridge tree is rooted at u. Our proof involves the following two claims.

- The sequence of ridge edges in the boundary of $source(a)$ (and hence, the ridge triangles in $source(a)$) have a one to one correspondence to the complete tree traversal of the ridge tree.
- The sequence of ridge edges in the boundary of $source(a_{(uv)})$ (and hence, the ridge triangles in $source(a_{(uv)})$) have a one to one correspondence to the portion from u to v of the tree traversal of the ridge tree that is rooted at u.
- The removal of duplicated ridge triangles in $source(a_{(uv)})$ is equivalent to removal of the duplicated edge traversals while traversing the ridge tree from u to v.
- The sequence τ'_{uv} of ridge triangles that are left over in τ_{uv} after removing the duplicates is the same as the path in the ridge tree from u to v, that is ρ_{uv}.

Thus we have $\rho_{uv} = r_1 \ldots r_k$. □

Lemma 5. *The back-bone of* $source(a_{(uv)})$ *is a convex polygon.*

Proof. Let $\tau'_{uv} = t_1 \ldots t_k$ be the sequence of ridge triangles in the back-bone of $source(a_{uv})$ as defined in the Lemma 4. We claim that the back-bone is a convex polygon for the following four reasons.

- Adjacent triangles t_i and t_{i+1} do not intersect except along their common side an_i.
- The angle at a is less than π for the following reason. Folding consists of only the triangles $t_1, t_2, ..., t_k$ and not their images $t'_1, t'_2, ..., t'_k$. Each t_i that is incident at a has its image t'_i also incident at a, and t_i and t'_i are similar triangles for each i. Since the angle at a is at most 2π in a convex polyhedron, the angle sum at a of the triangles $t_1, t_2, ..., t_k$ is at most π.

– The vertices of the Compact Source are $a, n_0 = u, n_1, n_2, ..., n_k = v$ as defined above. Since t_1 and t_k are triangles, the internal angles at n_0 and n_k are less than π. Consider any other $n_i, 0 < i < k$. Each such n_i is a ridge node. The internal angle at n_i in the back-bone is the sum of the internal angles at n_i in the triangles t_i and t_{i+1}. We will show in the next paragraph that the four triangles t_i, t'_i, t_{i+1} and t'_{i+1} that share the vertex n_i can together make an angle $< 2\pi$ at n_i. Since the sum of the angles at n_i in t_i and t_{i+1} is identical to that in t'_i and t'_{i+1}, the angle at each n_i is less than π in the back-bone. We will now show that four triangles t_i, t'_i, t_{i+1} and t'_{i+1} can together make an angle $< 2\pi$ at n_i. We have the following two cases.
  - n_i is not a polyhedral vertex: Then there are at least three shortest paths to that n_i from a. Hence, there are at least six triangles incident at n_i. Thus the four triangles t_i, t'_i, t_{i+1} and t'_{i+1} that share the vertex n_i can together make an angle $< 2\pi$ at n_i.
  - n_i is a polyhedral vertex: Then due to the curvature at n_i, the total angle at n_i is $< 2\pi$. Thus, the four triangles t_i, t'_i, t_{i+1} and t'_{i+1} that share the vertex n_i can together make an angle $< 2\pi$ at n_i.

Since the back-bone is a simple polygon with each internal angle $< \pi$, it is convex.

Proof of Theorem 1.

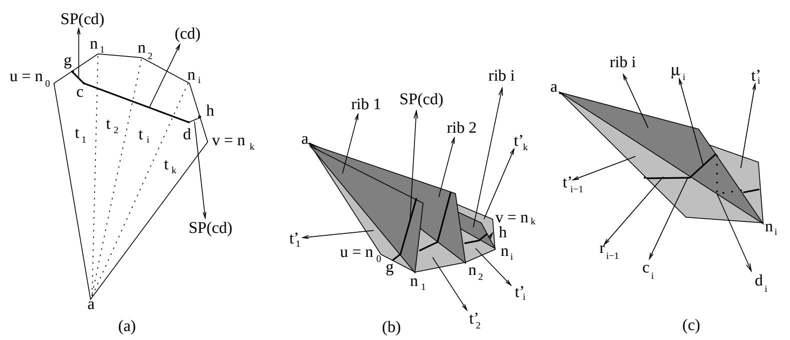


Fig. 12. Geodesics on a peel are unique shortest paths

See Fig.12(a). Let c and d be any pair of points on a peel p_1 in $source(a)$ of the polyhedron P. Since a peel is a convex polygon (see [7] [10]), there exists a geodesic (cd) joining c and d that lies entirely on the peel p_1. Contrary to the lemma, suppose that this geodesic (cd) is not a shortest path. Then we have the following two cases to consider.

Case 1: The polyhedron is degenerate with only three vertices a**,** c **and** d**:** In this case we have just two peels p_1 and p_2 in the polyhedron. The peel p_1

consists of exactly one ridge triangle t_1 whose vertices are a, c and d. The image t'_1 of t_1 is the peel p_2 itself with the same three vertices. Thus we have a unique path from c to d (that is a ridge edge as well) that is the boundary between the peels p_1 and p_2.

Case 2: Any other situation: Since we have assumed that (cd) is not a shortest path, we should have a shortest path $SP(cd)$ between c and d that is shorter than (cd) and that does not lie entirely on the peel p_1. Let u and v be the vertices of the polyhedron that lie on the peel p_1. Let $t_1 \dots t_k$ be the sequence of ridge triangles in the peel p_1 (see Fig.12(a)).

Consider the $source(a_{(uv)})$, its back-bone and ribs. The path in the ridge tree between u and v is the boundary of the peel p_1 between the two vertices u and v. Thus, the back-bone of $source(a_{(uv)})$ is formed by the sequence $t_1 \dots t_k$ with their images pasted on their back. Let $\{n_0 = u, n_1, n_2, ..., n_{s-1}, n_k = v\}$ be the sequence of ridge nodes in the ridge tree between u and v where $n_{i-1}n_i$ is a ridge edge r_i for each i. By definition each $n_i, 0 < i < k$, is a ridge node that is not a polyhedral vertex. Thus, there are at least six ridge triangles incident at each $n_i, 0 < i < k$. Since the ridge triangles t_{i-1} and t_i are adjacent along the edge an_i on the peel p_1 for each $i, 0 < i < k$, the triangles t'_{i-1} and t'_i cannot have a common edge an_i in the polyhedron. Thus, we have a rib (call it rib i) attached to the back-bone between the images t'_i and t'_{i+1} for each $i, 1 \le i < k$ in the Compact Source of $source(a_{(uv)})$ (see Fig.12(b)).

Since we have assumed that the shortest path $SP(cd)$ does not lie on the peel p_1, it should lie on multiple peels. Thus, we can partition $SP(cd)$ into maximal connected components such that each maximal connected component lies on a single peel in the Source Unfolding. These components are of the following types.

- *Components on $t_1 \dots t_k$ or equivalently, on $source(a_{(uv)}) = p_1$:* This includes the first connected component cg and the last connected component hd on the peel p_1 and possibly other intermediate components if any.
- *One or more connected components on $t'_1 \dots t'_k$ or equivalently, on the back-bone of $source(a_{(vu)})$ which is the compliment of $source(a_{(uv)}) = p_1$.*
- *One or more connected components on the ribs of $source(a_{(vu)})$.*

Our proof involves the following major steps. We will replace $SP(cd)$ by a continuous curve C that lies entirely on the peel p_1 and that is *not* longer than $SP(cd)$, that is we will have $C \le SP(cd) < (cd)$ (the latter inequality is our assumption at the beginning of this proof). However, any continuous curve on the peel p_1 from c to d cannot be shorter than the straight line segment (cd). Therefore, we have the straight line segment (cd) on the peel p_1 shorter than C, that is $(cd) < C$. But, this contradicts our claim that we have $C \le SP(cd) < (cd)$. Thus, we conclude that (cd) is the unique shortest path.

It is now sufficient to shown that we get a curve C that lies entirely on the peel p_1 and that is not longer that $SP(cd)$. Thus, it is done in the following steps.

- Step 1: We retain the connected components on $t_1 \dots t_k$ as they are. Hence, this step does not contribute to any change in the length of the resulting curve (see Fig.12(a)).

- We replace the connected component of $SP(cd)$ on each t'_i by an identical curve (of the same length) on its image t_i. This is possible because t'_i and t_i are identical. Hence, this step does not contribute to any change in the length of the resulting curve (see Fig.12(b)).
- Let μ_i denote the portion of $SP(cd)$ that lies on a rib between t'_{i-1} and t'_i for each $i, 1 < i < k$ (see Fig.12(c)). Let c_i and d_i be the end points of μ_i. Then c_i lies on the boundary an_i of t'_{i-1}. Similarly, d_i lies on the boundary an_i of t'_i. Thus, both c_i and d_i lie on the common edge an_i in the Compact Source that is shared by the ridge triangles t_{i-1}, t_i, t'_{i-1} and t'_i in addition to the ridge triangles that are in the rib i that contains μ_i. By triangle inequality, μ_i is longer than the straight line segment $c_i d_i$ that is a portion of an_i. Let us replace μ_i by the straight line segment $(c_i d_i)$ that is a portion of an_i. Since $(c_i d_i)$ lies on the common boundary an_i in the Compact Source, we can view $(c_i d_i)$ as a straight line segment on the peel p_1 itself. Hence, this step reduces the length of the resulting curve.

Note that the resulting curve C after the above substitutions on $SP(cd)$ is continuous and lies entirely on the peel p_1, and it is not longer that $SP(cd)$. □

Theorem 1 has the following consequences.

Proof of Corollary 1.

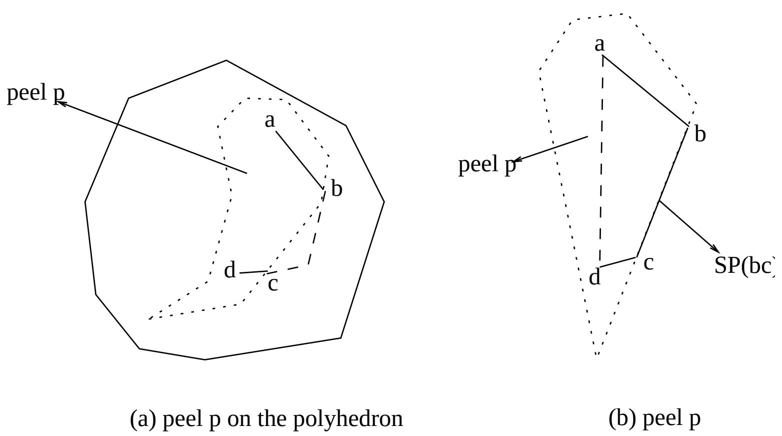


Fig. 13. Intersection of a shortest path with a peel

Contrary to the lemma, suppose that the intersection of a shortest path S with a peel p is not connected. Without loss of generality we assume that ab and cd are maximal connected components of the shortest path S that lie on the peel p and the portion bc of S lies outside the peel p as shown in Fig.13(a). The portion bc of S is a shortest path because S itself is a shortest path. However, according to Theorem 1, since b and c are on the same peel p, there is a unique shortest path bc between them on the same peel p as shown in Fig.13(b) and hence, we cannot have one more shortest path that is a portion of S outside the peel S, a contradiction. Hence, the intersection of a shortest path with a peel is connected. □

Proof of Theorem 2.

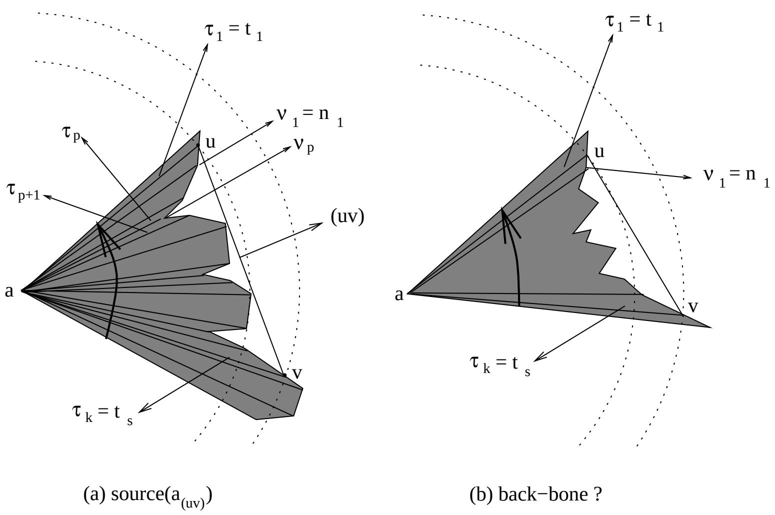


Fig. 14. $source(a_{(uv)})$ and its back-bone

Contrary to the theorem, suppose that there is no shortest path $SP(au)$ on any one of the ridge triangles in the sequence $\tau_2 \ldots \tau_k$. Then we will show that the back-bone of $source(a_{(uv)})$ is not a convex polygon. However, this contradicts Lemma 5, according to which back-bone of a $source(a_{(uv)})$ for any choice of a, u and v is a convex polygon. Thus, the theorem holds true. Now we will begin with an assumption that we do not have a shortest path $SP(au)$ on any one of the ridge triangles in the sequence $\tau_2 \ldots \tau_k$ and show that the back-bone of $source(a_{(uv)})$ is not a convex polygon.

Recall how we got the back-bone of $source(a_{(uv)})$. Since u and v are ridge points, we have a unique path ρ_{uv} from u to v on the ridge tree. ρ_{uv} contains a sequence of line segments $r_1 \ldots r_s$ where all except possibly r_1 and r_s are complete ridge edges while r_1 and r_s are portions of ridge edges ending at u and v respectively. We continue to call each line segment in $r_1 \ldots r_s$ a ridge edge for the sake of convenience. Let $t_1 \ldots t_s$ denote the sequence of triangles that corresponds to the sequence $r_1 \ldots r_s$ such that r_i is one of the sides of the triangle t_i for each i. We continue to call each triangle in the sequence $t_1 \ldots t_s$ a ridge triangle for the sake of convenience. The union of the triangles in the sequence $t_1 \ldots t_s$ is the back-bone of $source(a_{(uv)})$.

Let ν_i be the ridge node shared between the ridge triangles τ_i and τ_{i+1} for each $i, 1 \leq i < k$ in the $source(a)$. Let n_i be the ridge node shared between the ridge triangles t_i and t_{i+1} for each $i, 1 \leq i < s$, in the back-bone.

Our proof involves the following major claims (see Fig.14 (a) and (b)).

- **Claim 1:** The ridge node ν_1 in $source(a)$ is the same as the ridge node n_1 in the back-bone. Further, n_1 is a polygonal vertex of the back-bone (back-bone is a polygon).
- **Claim 2:** The points a, u, and v are polygonal vertices of the back-bone (back-bone is a polygon).
- **Claim 3:** The vertex n_1 in the back-bone lies inside the triangle formed by the vertices a, u, and v in the back-bone. A convex polygon cannot have one of its vertices inside the triangle formed by any other three vertices. Thus, the back-bone is not convex.

We will now prove these three claims.

Claim 1: *$\nu_1 = n_1$:* We have assumed at the beginning of this proof that we do not have a shortest path $SP(av)$ on any one of the ridge triangles in the sequence $\tau_2 \ldots \tau_k$. Thus, none of the triangle in the sequence $\tau_2 \ldots \tau_k$ is an image of τ_1. Thus, the back-bone will have τ_1 as its first triangle, that is $\tau_1 = t_1$. Since, ν_1 is one of the vertices of τ_1, we have $\nu_1 = n_1$ as a vertex of the back-bone (viewed as a polygon).

Claim 2: *a, u, and v are polygonal vertices in the back-bone:* Since $\tau_1 = t_1$, the vertices a and u of τ_1 will continue to be vertices in the back-bone. Since we have assumed that $SP(av) \geq SP(au)$, the $\angle vau < \pi$ and uv is outside $source(a)$ everywhere except at its end points, there does not exist a point q on $source(a_{(uv)})$ such that $SP(aq) \geq SP(av)$. Thus, we cannot have another shortest path $SP(av)$ in $source(a_{(uv)})$. Hence, we cannot have the image of τ_k in the sequence $\tau_1 \ldots \tau_{k-1}$ as well. Thus, τ_k belongs to the back-bone as the last triangle, that is we have $\tau_k = t_s$. Thus, the vertex v of the ridge triangle τ_k is a vertex in the back-bone as well. Thus a, u, and v are polygonal vertices in the back-bone. Let us denote the triangles formed by the three points v, a and u in $source(a_{(uv)})$ and its back-bone as $\triangle vau_{(source)}$ and $\triangle vau_{(backbone)}$ respectively.

Claim 3: *n_1 is inside the triangle $\triangle vau_{(backbone)}$:* Let $\angle vau_{(source)}$ and $\angle vau_{(backbone)}$ denote the angle at the vertex a in the triangles $\triangle vau_{(source)}$ and $\triangle vau_{(backbone)}$ respectively. Then $\angle vau_{(source)}$ is the sum of the angles at a in the triangles $\tau_1 \ldots \tau_k$, whereas $\angle vau_{(backbone)}$ is the sum of the angles at a in the triangles $t_1 \ldots t_s$. We will show in the last paragraph that $t_1 \ldots t_s$ is a proper subsequence of $\tau_1 \ldots \tau_k$. Thus, we have $\angle vau_{(backbone)} < \angle vau_{(source)}$. Since $\tau_1 = t_1$ is the first triangle in each of the structures $source(a_{(uv)})$ and its back-bone, its contribution to each of the angles $\angle vau_{(backbone)}$ and $\angle vau_{(source)}$ is the same. Therefore we can compare the contributions of the rest of the triangles to get the relation $\angle van_{1(backbone)} < \angle va\nu_{1(source)}$. Note that $\angle van_{1(backbone)} > 0$ because the triangle $\tau_k = t_s$ contributes to this angle. We will show in the next paragraph that n_1 lies inside the triangle $\triangle vau_{(backbone)}$ because the following four conditions are satisfied.

- $\angle va\nu_{1(source)} > \angle van_{1(backbone)} > 0$,
- $SP(av) \geq SP(au)$,
- $\tau_1 = t_1$ and
- $SP(av)$ is the same in both the $source(a_{(uv)})$ and its back-bone.

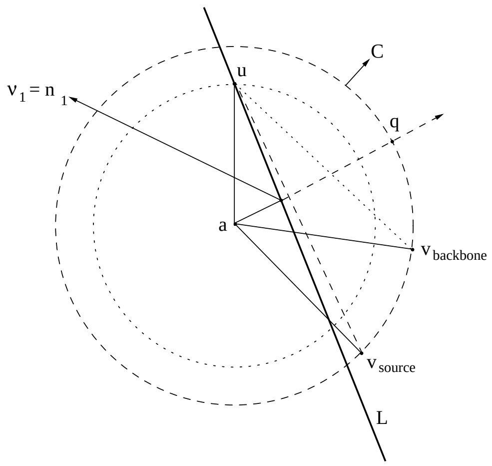


Fig. 15. super-imposition of $source(a_{(uv)})$ and its back-bone

Let us superimpose the $source(a_{(uv)})$ and its back-bone so that the triangles $\tau_1 = \triangle au\nu_1$ and $t_1 = \triangle aun_1$ coincide (see Fig.15). This is possible because $\tau_1 = t_1$. We shall call the point v in the $source(a_{(uv)})$ and its back-bone as $v_{(source)}$ and $v_{(backbone)}$ respectively. We draw the circle C with center at a and radius $SP(av)$. Both $v_{(source)}$ and $v_{(backbone)}$ lie on the circle C because $SP(av)$ is the same in both the $source(a_{(uv)})$ and its back-bone. Extend the line segment an_1 to meet the circle C at q. Let C' denote the portion of the circle C that we encounter when we move anti-clockwise from $v_{(source)}$ to q. Since we have $\angle va\nu_{1(source)} > \angle van_{1(backbone)} > 0$, we can have $v_{(backbone)}$ only on the portion C' of the circle C. Since we have assumed that $SP(av) \geq SP(au)$, there does not exist a point w that lies inside the triangle $\triangle vau$ such that $aw = SP(av)$. Thus for each possible position for $v_{(backbone)}$ on C', we have n_1 inside the triangle $\triangle vau_{(backbone)}$ as well.

$t_1 \ldots t_s$ is a proper subsequence of $\tau_1 \ldots \tau_k$: According to the definition of back-bone, $t_1 \ldots t_s$ is a subsequence of $\tau_1 \ldots \tau_k$. Thus, it is sufficient to show that some of the ridge triangles in $\tau_1 \ldots \tau_k$ do not belong to $t_1 \ldots t_s$, so that $t_1 \ldots t_s$ is a proper subsequence of $\tau_1 \ldots \tau_k$. Consider the two ridge triangles τ_1 and τ_k. We claim that τ_1 and τ_k belong to two different peels for the following reason. Contrary to the claim, suppose that τ_1 and τ_k belong to the same peel. Then we will have both the points u and v on this peel and hence, the line

segment (uv) lies on this peel. This contradicts our assumption that (uv) lies outside $source(a)$ except for its end points. Hence, we conclude that τ_1 and τ_k lie on different peels. Thus, at least one of the ridge nodes say an $\nu_p, 1 \leq p < k$, is a polyhedral vertex. Then the two ridge triangles τ_p and τ_{p+1} are images of each other, and hence, they belong to a rib of $source(a_{(uv)})$. Hence, τ_p and τ_{p+1} belong to $\tau_1 \dots \tau_k$ but not to $t_1 \dots t_s$. Hence, $t_1 \dots t_s$ is a proper subsequence of $\tau_1 \dots \tau_k$. □

9 Star Unfoldings: Proof of Theorem 3

Proof. There are three cases, depending upon whether ab lies completely inside the Star Unfolding, completely outside the Star Unfolding, or partly inside and partly outside the Star Unfolding.

Case 1. If ab lies completely inside the Star Unfolding then, by Fact 1, this line segment is an image of a geodesic on the surface of $\mathcal{P}$ (see Fig.1(b)). The length of this geodesic is clearly at least $SP(ab)$. Theorem 3 holds, therefore, for this case.

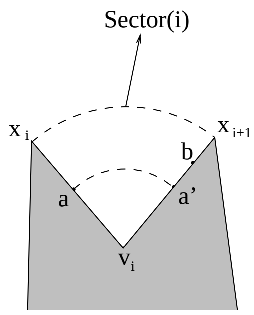


Fig. 16. Case 2: Both a and b on the same sector

Case 2. Next, suppose ab lies completely outside the Star Unfolding. Clearly, both a and b must lie on the boundary of the Star Unfolding. Recall that, by Lemma 1, the Star Unfolding is a simple polygon. Let $Sector(i)$ and $Sector(j)$ be sectors containing a and b, respectively. There are two sub cases in this case, depending upon whether or not $i = j$.

First, suppose $i = j$. Then it is easily seen that ab is completely contained within $Sector(i)$ (see Fig.16). Recall that $Sector(i)$ is a sector of the circle centered at v_i and is bounded by the two radii $x_i v_i$ and $x_{i+1} v_i$. Since, by Lemma 1, this sector is completely outside the Star Unfolding, both a and b lie on the boundary of the $Sector(i)$. Since the boundary of the $Sector(i)$ consists of the two radii $x_i v_i$ and $x_{i+1} v_i$, and ab is assumed to be outside the Star Unfolding, we assume without loss of generality that a must be on $x_i v_i$ and b on $x_{i+1} v_i$. Further, without loss of generality, $star(v_i a) \leq star(v_i b)$. Let a' be the unique point on $x_{i+1} v_i$ such that a and a' are images of the same point on $\mathcal{P}$. We claim that $a'b < ab$. Consider the two triangles $\triangle a v_i b$ and $\triangle a' v_i b$. The angle $\theta = \angle a v_i b > \angle a' v_i b = 0$, where θ is the curvature of the polyhedron at v_i. Further, $v_i a = v_i a'$ and $v_i b$ is common to both the triangles. Hence, we have $a'b < ab$. That is, we have $star(a'b) \leq star(ab)$. Since $a'b$ is completely inside the Star Unfolding, by Case 1, it follows that $SP(a'b) \leq star(a'b)$. Finally, since a' and a correspond to the same point on $\mathcal{P}$, we have $SP(ab) \leq SP(a'b)$. Thus we have $SP(ab) \leq SP(a'b) \leq star(a'b) \leq star(ab)$, and the theorem follows in this case.

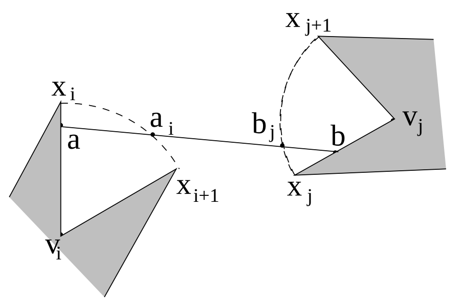


Fig. 17. Case 2: Both a and b on the same sector

Second, suppose $i \neq j$ (Fig.17). Note that a must lie on one of $v_i x_i$ or $v_i x_{i+1}$; similarly, b must lie on one of $v_j x_j$ or $v_j x_{j+1}$. Without loss of generality, we assume that a lies on $v_i x_i$ and b on $v_j x_j$ (the other cases are identical). We show in the next paragraph that $star(ax_i) + star(x_j b) \leq star(ab)$. Note that x_i and x_j are images of the same point x on $\mathcal{P}$. Then, since $ax_i, x_j b$ are both in the interior of the Star Unfolding, using Case 1, there exists a path from a to b through x on $\mathcal{P}$ which is no longer than $star(ax_i) + star(x_j b) \leq star(ab)$. It follows that $star(ab) \geq SP(ab)$.

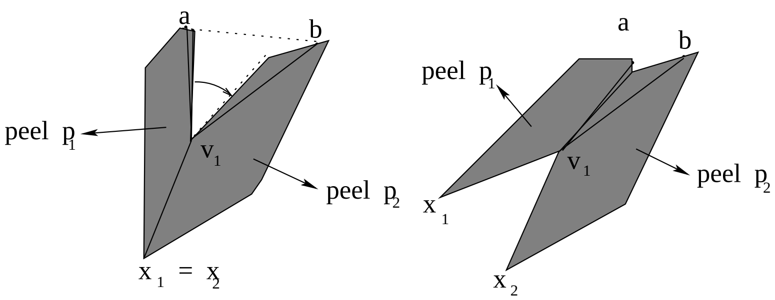
We need to show that $star(ax_i) + star(x_j b) \leq star(ab)$. Recall from Lemma 1 that $Sector(i)$ and $Sector(j)$ do not overlap. Let aa_i be the maximal portion of ab within $Sector(i)$ and let $b_j b$ be the maximal portion of ab within $Sector(j)$; then aa_i and $b_j b$ are disjoint line subsegments of ab. So $star(aa_i) + star(b_j b) \leq star(ab)$. Further, $star(aa_i) \geq star(ax_i)$ and $star(b_j b) > star(x_j b)$ (from any point inside a circle, the nearest point on the circumference is the one which lies on the straight line joining this point to the center). The claim follows.

Case 3. Suppose ab doesn't lie completely in the interior or completely in the the exterior of the Star Unfolding. In this case, we decompose ab into several (finitely many) subsegments, each subsegment lying completely in the interior or completely in the exterior. By cases 1 and 2, we know that the theorem holds for each subsegment. The theorem follows immediately for the whole segment as well. □

10 Source Unfoldings: Proof of Theorem 4

Recall that Fact 1 holds for Source Unfoldings as well. So, as in the proof of Theorem 3, if ab is completely inside the Source Unfolding then Theorem 4 follows immediately. The difficult case is when ab lies completely outside the Source Unfolding. The rest of this section will address this case. Once Theorem 4 is shown for this case, it will follow for the more general case (i.e., ab being partly inside and partly outside) as well, exactly as in the proof of Theorem 3.

For the rest of this section, assume that ab lies completely outside the Source Unfolding. Both a and b must then be on the boundary of the unfolding. We will show that the following Lemma 6 holds in this case. Theorem 4 then follows from Lemma 6.



(a) peels as in Source Unfolding (b) peels as in Star Unfolding

Fig. 18. $source(ab) \geq SP(ab)$

Lemma 6. *If a, b are two points on the boundary of the Source Unfolding such that ab is completely outside the unfolding, then $source(ab) \geq SP(ab)$.*

Proof: Consider the triangle $\triangle axb$ in the Source Unfolding and consider all the peels in the Source Unfolding which intersect this triangle. Without loss of generality, let the peels which intersect this triangle be $p_1 \ldots p_k$, for some $2 \leq k < m$ (m is the total number of peels in the Star Unfolding). The proof of Lemma 6 is by induction on k. Note that $k \geq 2$, otherwise (that is, when $k = 1$)

we have both a and b on the same peel; therefore ab lies entirely on a single peel p_1 and hence, ab cannot lie outside the Source Unfolding at all.

basis: $k = 2$: Then we have a and b on the peels p_1 and p_2 respectively, v_1 is the vertex common to the peels p_1 and p_2 and r_1 is the ridge edge that is incident at v_1. Note that the three line segments v_1a, r_1 and v_1b are in the clockwise order around v_1. Let us cut the peel p_1 of the Source Unfolding along xv_1 and paste the peels p_1 and p_2 along the ridge edge r_1. This is the same as rotating the peel p_1 with respect to the peel p_2 as in the Source Unfolding to its position as in Star Unfolding with respect to the peel p_2 as shown by the arrow in Fig.18. Thus when we move the peel p_1 from its old position as in Source Unfolding to its new position as in Star Unfolding, v_1a turns clockwise around v_1 by the same angle as is the curvature at v_1. Since the ridge edge incident at v_1 bisects the two copies of x on the peels p_1 and p_2 in Star Unfolding, the peels p_1 and p_2 lie entirely on the opposite sides of r_1, and hence the clockwise order of v_1a, r_1 and v_1b is preserved after rotation. Since ab is outside the Source Unfolding, whereas v_1a and v_1b are inside, the angle $\angle av_1b$ in the Source Unfolding is $< \pi$. Thus, as a consequence of the rotation, the angle $\angle av_1b$ is smaller in the Star Unfolding when compared to the same angle in the Source Unfolding. Now compare the two triangles $\triangle v_1ab$ in Source Unfolding and Star Unfolding. Since the two sides v_1a and v_1b are the same in both the triangles whereas the angle $\angle av_1b$ is smaller in the Star Unfolding, the third side ab is smaller in the Star Unfolding. But the side ab in these triangles with respect to Source Unfolding and Star Unfolding are nothing but $source(ab)$ and $star(ab)$ respectively. Hence, we have $source(ab) \geq star(ab) \geq SP(ab)$ in this case (the latter inequality due to Theorem 3).

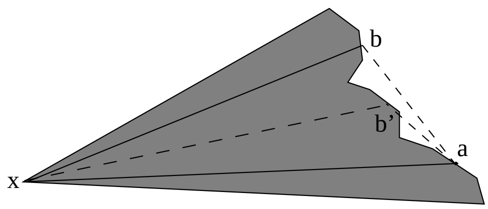


Fig. 19. Existence of another copy b' of b

Induction: Suppose we have proved the lemma for all $i < k$ for some $k > 2$. Then we will show that the lemma holds for k as well, and it is as follows. Suppose we have the points a and b on the peels p_1 and p_k respectively. Without loss of generality, we assume that the shortest path $SP(xa)$ on the peel p_1 is at least as large as the shortest path $SP(xb)$ on the peel p_k. Since we assume that ab lies completely outside the Star Unfolding, according to Theorem 2 in Section 3, we have another shortest path $SP(xb)$ on a peel p_i, $1 \leq i < k$. Let us call this copy of b as b' (see Fig.19). We will show in the next two paragraphs that $source(ab) > source(ab')$ and $source(ab') \geq SP(ab)$. That would complete the proof of Lemma 6.

$source(ab) > source(ab')$: Consider the two triangles $\triangle axb$ and $\triangle axb'$ (see Fig.19). We have the following.

- Since ab is outside the Star Unfolding, the angle $\angle axb < \pi$.
- Since the peel p_i is completely inside the triangle $\triangle axb$, the angle $\angle axb' < \angle axb$.
- The side ax is common to both the triangles.
- Since xb' lies on a single peel, it is a shortest path and hence, $source(xb) = source(xb')$.

Thus, we have $source(ab) > source(ab')$.

$source(ab') \geq SP(ab)$: $source(ab')$ can be split into maximal connected components that are completely on the Source Unfolding and that are completely outside the Source Unfolding. Consider any connected component, say cd. If cd lies completely on the Source Unfolding, then $cd \geq SP(cd)$ according to Fact 1. Suppose cd is completely outside the Source Unfolding. Then cd must span fewer peels than k because $i < k$. By the induction hypothesis, $cd \geq SP(cd)$. Thus, $source(ab') > SP(ab') = SP(ab)$. □

11 Planar Unfoldings can have Short-Cuts

Consider the Planar Unfolding with respect to a sequence of faces $f_1 \dots f_k$ as defined in the introduction. As mentioned earlier, Planar Unfoldings could have short-cuts, i.e., there could be points (x, y) in the Planar Unfolding such that $planar(xy) \leq SP(xy)$. We provide an example for a short-cut in a Planar Unfolding below. It may be noted that an example for the overlap of Planar Unfolding is shown in [5]. Overlap of planar unfolding is a special case of a short-cut where the distance between the overlapping points x and y is equal to zero.

Let us construct a convex polyhedron whose unfolding has a short-cut (x, y) as follows. See Fig.20(a). The shaded portion is a 'tetrahedral cap' on a larger convex polyhedron. The tetrahedral portion has the edges e_1, e_2, e_3 and e_4 and three faces f_1, f_2 and f_3. We assume that the curvature at the vertex v_3 on the tetrahedron is very small. We assume that the rest of the polyhedron (excluding the tetrahedron) is very large so that we have a unique shortest path $G = xy$ from a point x on the edge e_1 to a point y on the face f_3 as shown in the Fig.20(a). xy needs to be a shortest path because, we need the corresponding edge sequence to be a shortest path edge sequence. The shortest path G intersects the edges e_1, e_2 and e_4. We also assume that x is arbitrarily close to v_2 and y is arbitrarily close to the edge e_4. Let us now cut this polyhedron as follows.

- First, we cut the polyhedron along the plane through the three points v_3, v_1, y. We get a new edge e_5 which passes through the points v_3 and y as shown in the Fig.20(b). The geodesic G that was representing the shortest path from

x to y before cutting remains unaltered in the new solid because, we are cutting the polyhedron along v_3y and v_3v_1. However, G need not be a shortest path in the new solid because, with a loss of a *slice* of the polyhedron due to the present cut, there can be a shorter path G' from x to y that traverses the new face created (as shown by a dotted curve in the Fig.20(b)). Since we can choose y as close to e_4 as we want, the present cut can be as thin a slice as we desire. With arbitrarily thin slice being cut, one can assume that the geodesic G is a shortest path in the new solid as well.

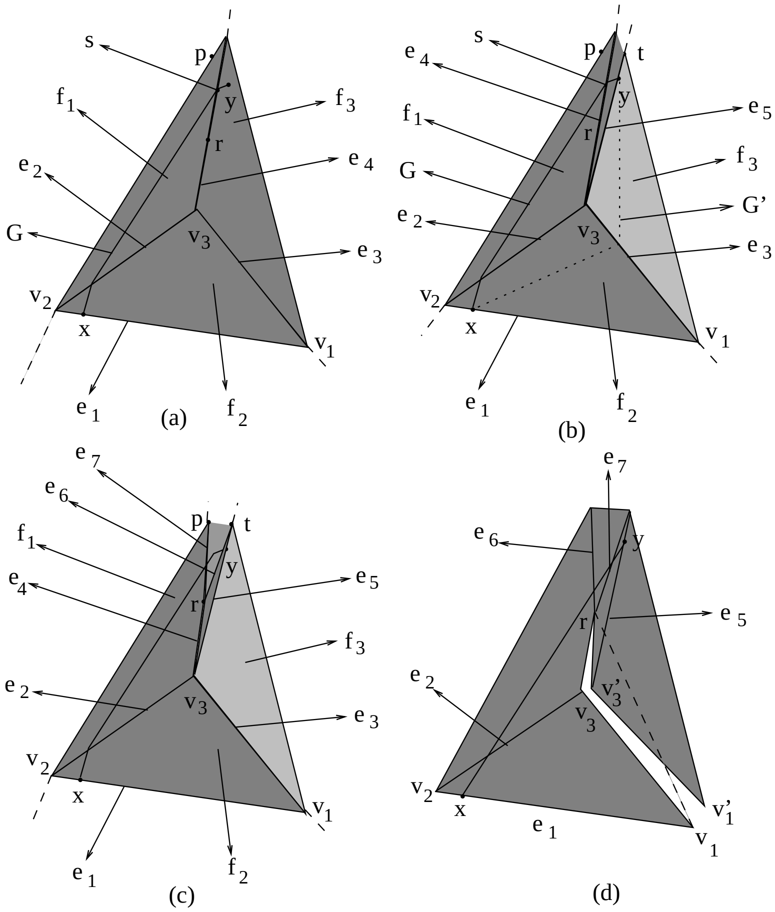


Fig. 20. An example for a short-cut

– Let the shortest path G cross the edge e_4 at s. Let r be any point on the edge e_4 between s and v_3. Let t be any point on the edge e_5 above y (that is, y lies on the portion v_3t of e_5). Now let us cut this new solid again along a plane that passes through r, t and p, where p lies on the face f_1 above G (see the Fig.20(b)). Let e_6 and e_7 be the new edges that we get due to the second cut (see Fig.20(c)). Further, as a consequence of the second cut, the edge e_4 is now restricted to its portion between v_3 and r. We claim that the shortest path from x to y in the new solid traverses the new face that lies between the new edges e_6 and e_7 for the following reason. Contrary to the claim suppose that the shortest path from x to y in the new solid does not traverse the new face that lies between the edges e_6 and e_7. Let G_1 denote this shortest path from x to y on the new solid. Then G_1 should lie on that portion of the new solid that was also part of the older solid (that is, before the present cut). Thus, G_1 was a geodesic on the earlier solid as well. Since G was the shortest path from x to y on the earlier solid, we have $G \leq G_1$. Now let us superimpose the earlier solid and the present one. Consider the curve G that was on the older solid in relation to the new solid. G lies on the new solid everywhere except for a connected portion between the new edges e_6 and e_7 where it is off the new solid. We can shrink this portion like an elastic band so that this portion now lies on the new solid. Thus, G shrinks to a new curve G_2 that completely lies on the new solid and hence, we have $G_2 < G$. Thus, we have $G_2 < G \leq G_1$. However, this leads us to the following contradiction. We have two curves G_1 and G_2 on the new solid between the points x and y where $G_2 < G_1$, but G_1 is the shortest path. Thus, our claim that the shortest path from x to y in the new solid traverses the new face that lies between the new edges e_6 and e_7 holds.

Let us now cut this new solid shown in the Fig.20(c) along e_3 and e_4 and unfold it on a plane as in Fig.20(d). Because of the curvature at r, we have a wedge shaped gap between two copies of the edge e_4. Note that v_3 and v_1 are duplicated in the unfolding. Let v_3 and v_3' be the two copies of v_3 and v_1 and v_1' be the two copies of v_1. We will show that $v_3'v_1 < SP(v_3v_1)$. Consider the three line segments rv_3, rv_3' and rv_1 in the unfolding shown in the Fig.20(d). When the curvature of the polyhedron at the vertices v_3 and r are sufficiently small, we have rv_3' between rv_3 and rv_1 (that is, we encounter rv_3' as we move anti-clockwise from rv_3 to rv_1). Consider the two triangles $\triangle rv_3v_1$ and $\triangle rv_3'v_1$. $rv_3' = rv_3$ because they are the copies of the same edge. $\angle v_3rv_1 > \angle v_3'rv_1$ because we have rv_3' between rv_3 and rv_1. Hence, we have $v_3v_1 > v_3'v_1$. But, v_3v_1 is a portion of an edge and hence, it is the unique shortest path between its end points. Thus, (v_3', v_1) is shorter than $SP(v_3v_1)$, and it is a short-cut. In general, for chosen curvatures at v_3 and r, we will have pairs of points (m, n) where m is in a small neighborhood of v_3', n is in a small neighborhood of v_1 and (m, n) is a short-cut.