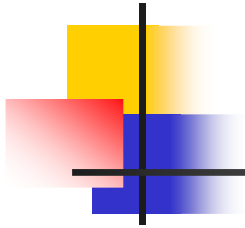




# Topics in Algorithms 2007

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# Tree Embeddings



# Projections

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- Can vertices in a given weighted graph  $G$  be mapped to vertices in any prob distribution over edge-weighted trees  $H$  so that all distances (i.e., shortest paths) only increase, but not by too much?
- Distortion: max stretch over all edges in  $G$



# Upper Bound

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- Fakcharoenphol, Rao, Talwar upper bound of  $\log n$  for probabilistic embeddings



# Upper Bound

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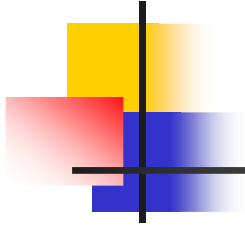
- Hierarchical cluster decomposition
  - Parameter  $r$  (radius) which shrinks with time
  - Each cluster  $C$  at time  $i$  splits into further clusters at time  $i+1$  as follows
  - Every vertex in  $C$  attaches to a center within distance  $r$
  - Each center gives a new sub cluster
  - The radius of each new sub cluster is at most  $r$



# Upper Bound

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- The cluster decomposition tree
  - Root: cluster with all vertices
  - Internal node: a cluster  $C$
  - Children: subclusters of  $C$
  - Weight of edge from  $C$  to its children:  $2r$  (diameter of  $C$ ), guarantees expansion
  - Leaf: individual nodes
  - What is the expected Distortion (over random choices)??



## Bounding Expected Distortion

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If vertices  $u$  and  $v$  get split when  $r=x$ ,  
and  $r$  shrinks geometrically then  
distortion is  $O(x)$

Danger:  $x \gg d(uv)$

Need to make prob of splitting  $uv$  small when  $r \gg d(uv)$



# Randomness

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- We need randomness to defeat the lower bound. Where does this randomness come from?
- How are centers chosen?
- Each vertex picks an arbitrary/random center within distance  $r$ ? Doesn't work because vertices close by can split very early on
- Flip around and consider centers one by one in random order, take all vertices within  $r$  of a center to create a new cluster
- Now if  $uv$  are close to each other than hopefully they will both attach to the same center and not split early





# Randomness

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- What happens if there is only one center  $w$  within  $r$  of  $u$  and  $v$  is just outside the reach of  $w$ ;
- Then irrespective on randomness in center choice,  $uv$  will split early
- So we need more randomness; introduce randomness in  $r$
- $r$  should have a distribution spread over a range that is proportional to its mean; for instance take  $r_0$  uniformly in  $[1,2]$  at the very end and  $r$  is  $r_0 \cdot 2^i$  where  $i$  runs back in time; so at time  $i$ ,  $r$  is uniform in  $2^i \dots 2^{i+1}$
- So if  $d(uv)$  is much smaller than  $r \sim d(wu), d(wv)$  then this randomness ensures that the chances of  $uv$  being split is down to  $d(uv)/r$



# Bounding Expected Distortion

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2 parameters

- Which vertex splits  $uv$  (if 2 vertices take the closer one)?
- At what  $r$  does the split happen?

$\sum_{\{x,w\}} x \cdot \text{Prob}(\text{split happens due to } w \text{ when } r=x..2x)$

- $\sum_{\{x,w\}} x \cdot 0$  if  $d(w,u), d(w,v)$  are both  $>r$  or both  $<r$
- $\leq \sum_{\{x,w\}} x \cdot \frac{1}{i}$  if  $d(w,u) >r$  and  $d(w,v) <r$  and  $w$  is the  $i$ th closest vertex to edge  $uv$
- $= \sum_{\{x,w\}} x \cdot \frac{1}{i} \cdot \frac{d(uv)}{x}$
- $\log n \cdot d(uv) !!$