Topics in Algorithms 2005
Linear Programming and Duality

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The Steiner Tree Problem

Given a subset H of steiner vertices which need to be connected together, find the least cost connection.

min Σ c_e x_e

for each cut S splitting H:

Σ_{e crossing S} x_e >= 1

for each edge e:

x_e>=0

Lagrangian Formulation

If there were no constraints, we could minimize via differentiation

How do we eliminate constraints? Move them into the objective function with penalties.

$\min_{x_e} \max_{a_{S},b_{e}}$

$\Sigma_{e} c_e x_e - \Sigma_{S \text{ splitting } H} a_S (\Sigma_{e \text{ crossing } S} x_e - 1) - \Sigma_e b_{e} x_e$

$a_{S},b_{e} >= 0$

Why is the lagrangian optimum the same as the LP optimum?

Lagrangian Dual

Just flip min and max order

$$\max_{a_{S},b_{e}} \min_{x_e}$$

$$\Sigma_{e} c_e x_e - \Sigma_{S \text{ splitting } H} a_S (\Sigma_{e \text{ crossing } S} x_e - 1) - \Sigma_e b_{e} x_e$$

$$a_{S},b_{e} >= 0$$

Why is the lagrangian dual optimum the same as the lagrangian primal optimum?

The LP Dual

Differentiate with respect to x_e as there are no constraints on x_e; this will eliminate x_e but yield some new constraints

$\max_{\{a_{S}, b_{e}\}} \Sigma_{S} a_{S}$

$c_e = \Sigma_{S, e \text{ crosses } S} a_{S} + b_{e}$

$a_{S}, b_{e} >= 0$

$\max_{\{a_{S}\}} \Sigma_{S} a_{S}$

$\Sigma_{S, e \text{ crosses } S} a_{S} <= c_e$

$a_{S} >= 0$

Why is the LP dual optimum same as the Lagrangian dual optimum?

It follows that LP primal and LP dual have the same optima

The Steiner Tree Problem: The Primal and Dual LPs

Given a subset S of vertices which need to be connected together, find the least cost connection.

$\min \Sigma\, c_e x_e$

for each cut splitting S:

$\Sigma\, x_e >= 1$

for each edge e:

$0 <= x_e$

$\max_{\{a_{\{S\}}\}}\ \Sigma_{\{S\}}\, a_{\{S\}}$

$\Sigma_{\{S, e \text{ crosses } S\}}\, a_{\{S\}} <= c_e$

$a_{\{S\}} >= 0$

Lagrangian Primal Equals Lagrangian Dual

Lagrangian dual is smaller than Lagrangian primal: Easy! Exercise.

Lagrangian dual equal to Lagrangian primal?? How is this shown?

The General Derivation

- Proof that appropriate Lagrange Multipliers always exist?

> - Roll all primal variables into w
>   lagrange multipliers into λ

$\min_{w} f(w)$ $Xw >= y$	$\min_{w} \max_{\lambda>=0} f(w) - \lambda (Xw-y)$	$\max_{\lambda>=0} \min_{w} f(w) - \lambda (Xw-y)$

The General Derivation

- Show that there exists λ such that minimizing
  f(w) – λ (Xw-y) over w yields f(w*) where w* is the primal optimum

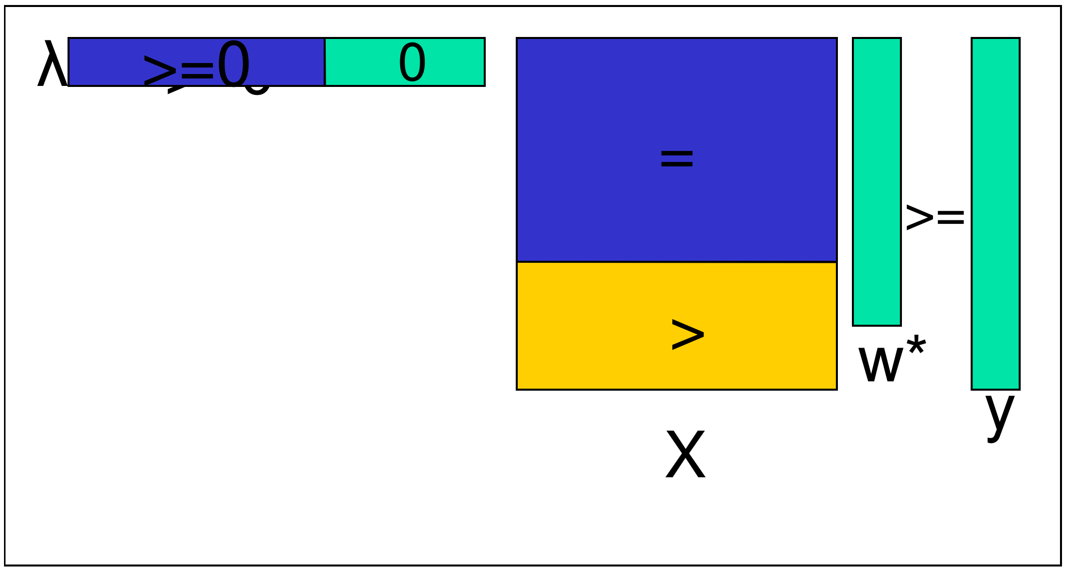
$$\min_{w} f(w) \quad Xw \geq y$$

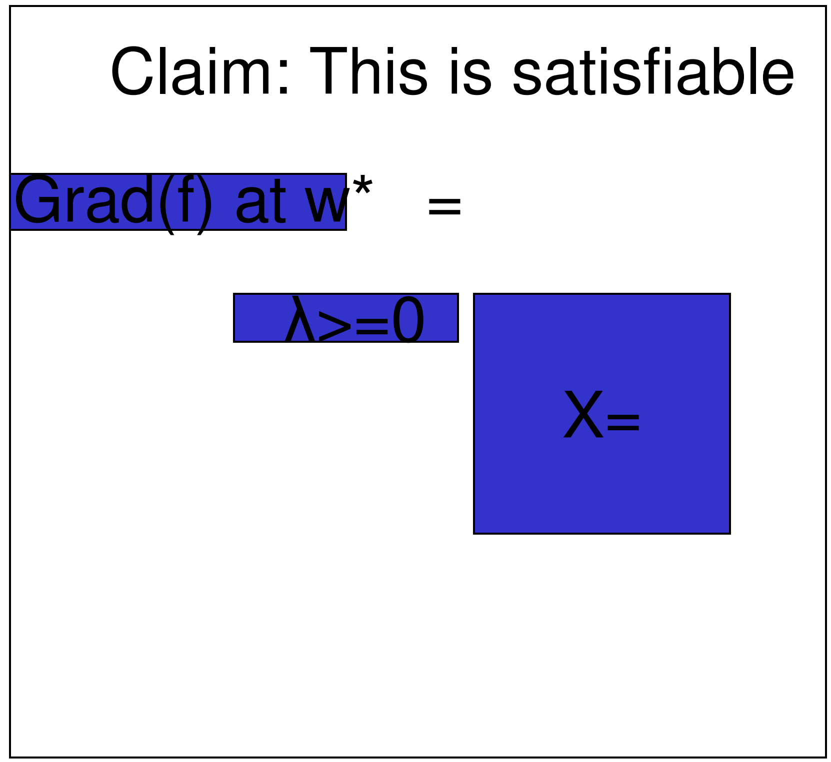
$$\min_{w} \max_{\lambda \geq 0} f(w) - \lambda (Xw-y)$$

$$\max_{\lambda \geq 0} \min_{w} f(w) - \lambda (Xw-y)$$

The General Derivation

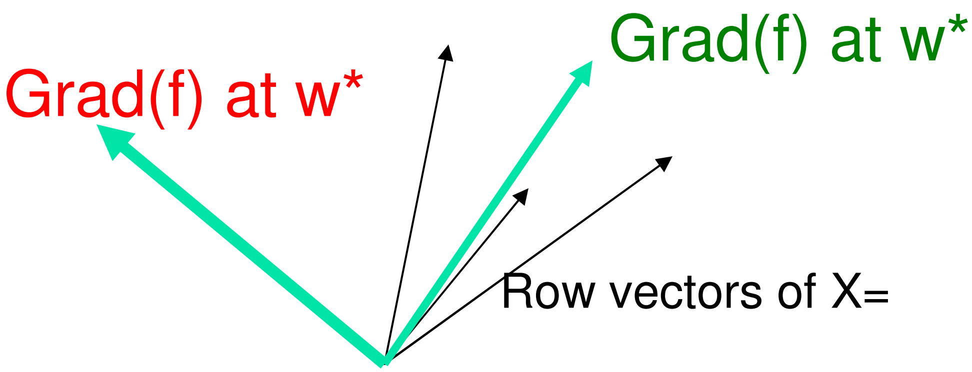
- Proof that there exists λ such that minimizing
  f(w) – λ (Xw-y) over w yields f(w*) where w* is the primal optimum





The General Derivation

- Grad(f) at w* should be in the cone..



Claim: This is satisfiable

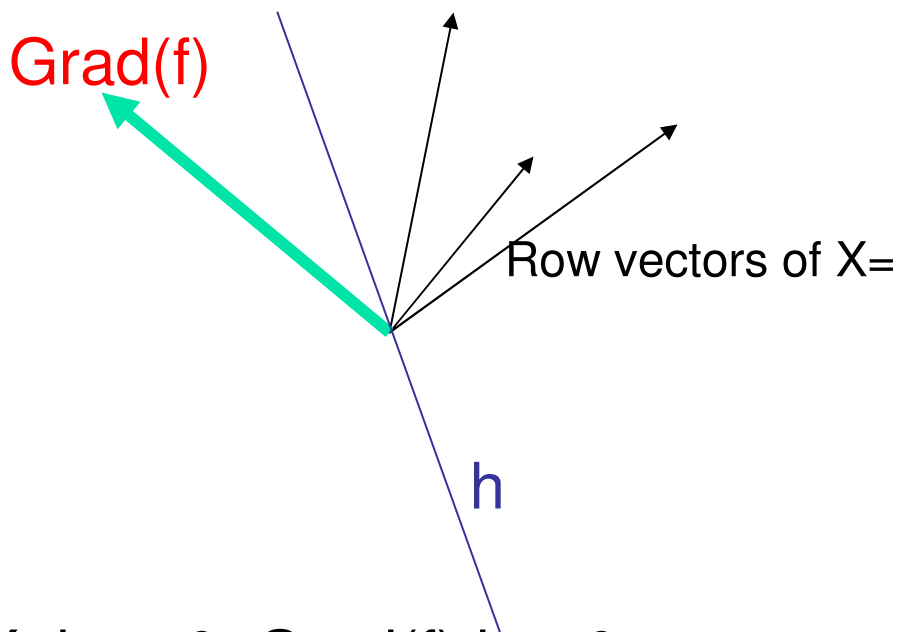
Grad(f) at w* =

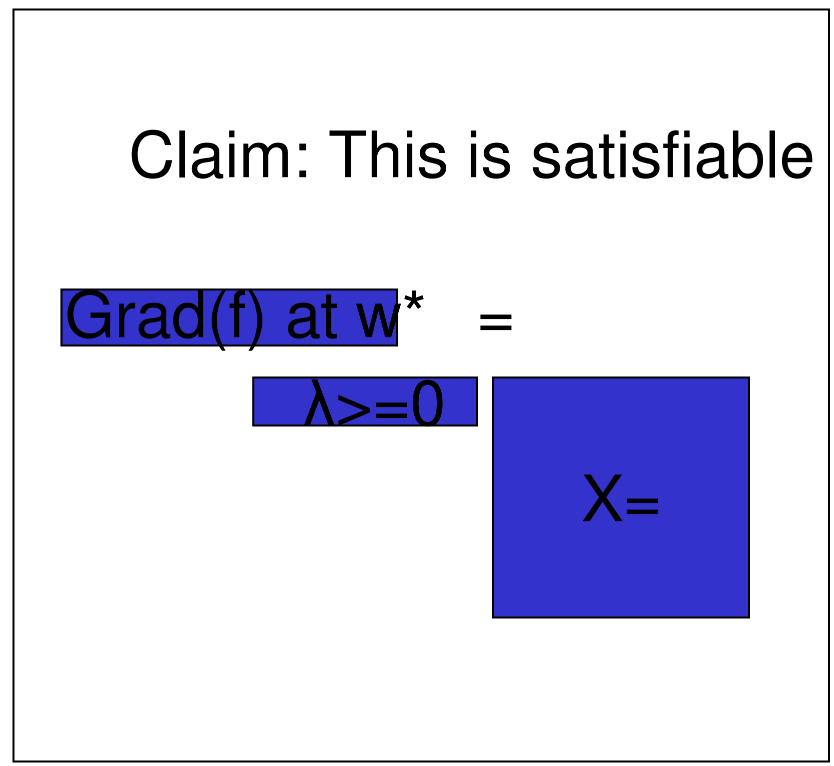
λ>=0

X=

The General Derivation

- If Grad(f) at w* is not in the cone, then use Farkas' Lemma





$X_= h >= 0$, Grad(f) $h < 0$

w*+h is feasible and f(w*+h)<f(w*) for small enough h

Complementary Slackness or Karush-Kuhn-Tucker conditions

- If w^*,λ^* are primal and dual solutions then they are optimum solutions if and only if the following are satisfied:

  - if a particular λ_i is non-zero, then the corresponding primal inequality is satisfied with equality

  - if a particular w_i is non-zero, then the corresponding dual inequality is satisfied with equality

The Primal Dual Approach

- Find feasible primal and dual solutions
- Dual Solution serves as a lower bound
- Primal Solution is integral and serves as the final answer

$\min \Sigma c_e x_e$

for each cut splitting S:

$\Sigma x_e >= 1$

for each edge e:

$0<=x_e$

$\max_{a_{S}} \Sigma_{S} a_{S}$

$\Sigma_{S,e \text{ crosses } S} a_{S} <= c_e$

$a_{S}>= 0$

References

- Kamal Jain's paper on the generalized steiner network problem
- Goemans and Williamson's paper on the Steiner Tree problem
- Goemans,Williamson,Vazirani,Mahail on the Steiner Network Problem

All available on the web