The Mahler Measure of a Polynomial

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The Setting

- A polynomial $f(x)$ with integral coefficients.
- We want a measure $M(f) : f -> \mathcal{R}$ of complexity which is related to (upper and lower bounded within some factors) the norm of f (which norm?).
- And which is multiplicative, i.e., $M(fg) = M(f)M(g)$

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Possibilities

- Use a norm (0,1,2 or ∞) itself as the measure? Why doesn't this work.
- The multiplicative property requires that the measure be somehow related to a product of roots.
- Any integer polynomial with degree d has d (possibly complex and repeated) roots. Why?
- The product of all roots (or their moduli) is not an option. Why?

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The Mahler Measure

- $f(x) = \sum_0^d a_k x^k = a_d \Pi_1^d (x - \alpha_i)$
- $M(f) = |a_d| \Pi_i \max\{1, |\alpha_i|\}$ where α_i's are roots of f and a_d is the coefficient of x^d.
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Relationship with the Norm

- $|a_k| \leq \binom{d}{k} M(f)$. Why?
- It follows that there are only finitely many polynomials with integer coefficients and degree *d* having Mahler Measure smaller than some specified number *m*. Why?
- Corollary: $||f||_2 \leq 2^d M(f)$. Why?

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Relationship with the Norm

- $M(f)^2 + |a_0 a_d|^2 M(f)^{-2} \leq ||f||_2^2$
- Hint: Consider the polynomial $g = a_d \Pi_{j=1}^{k} (\bar{z}_j x - 1) \Pi_{j=k+1}^{d} (x - z_j)$ where the latter product is over roots of f outside the unit circle and the former over roots of f inside the unit circle. Note $||f||_2^2 = ||g||_2^2 \geq M(f)^2 + |a_0 a_d|^2 M(f)^{-2}$.
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Jensen's Inequality

- $\log M(g) = 1/2\pi \times \int_0^{2\pi} \log|g(e^{it})|dt$
- Why?
- Use $g(e^{it}) = |f(e^{it})|^2$ and $e^{1/2\pi \times \int_0^{2\pi} log|g(e^{it})|dt} \leq 1/2\pi \times \int_0^{2\pi} |g(e^{it})|dt$ to get an alternative proof of $M(f) \leq ||f||_2$?

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Smith's Inequality

- Claim: The Mahler measure of an irreversible irreducible polynomial $f(x)$ with integral coefficients is at least 1.18.
- We can assume that $f(0) \neq 0$ and so $|a_d| = |a_0| = 1$. Why?

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- Consider the Blaschke function $B(x) = \Pi_{j=1}^{k} \frac{x-\alpha_j}{1-\bar{\alpha}_j x}$ where we take only roots of f inside the unit circle, and consider its Taylor expansion about 0, $c_0 + c_1 x + c_2 x^2 \ldots$
- Then $|B(x)| = 1$ if $|x| = 1$ and consequently, $1 = 1/2\pi \times \int_0^{2\pi} |B(e^{it})|^2 dt = \sum |c_i|^2 = 1$.
- Let f^* denote the reverse of f, i.e., $f^*(x) = x^d f(1/x)$. If f has a root on the unit circle then so does f^*, and a root of f inside the unit circle maps to a root of f^* outside the unit circle.
- Let B, B^* denote the Blaschke functions of f, f^*, respectively. Note $B/B^* = f/f^*$. Why?

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- Let c_k, d_k, a_k denote the coefficients of the Taylor expansion of $B, B^*, f/f*$ respectively.
- Since f is irreversible f/f^* is not a constant so there exists a smallest $l >= 1$ such that $a_l \neq 0$. In fact a_l must be integral and therefore $|a_l| \geq 1$.
- And $|c_0| = |d_0| = 1/M(f)$
- Since $B/B^* = f/f^*$, we have $c_l - a_0 d_l = a_l d_0$, so $c_l - d_l = a_l d_0$.
- So either $|c_l| \geq |a_l|/2M(f)$ or $|d_l| \geq |a_l|/2M(f)$.
- Since $\sum |c_i|^2 = \sum |d_i|^2 = 1$, it follows that $(1 + |a_l|^2/4)/M^2(f) \leq 1$.
- $M(f)^2 \geq (1 + |a_l|^2/4) \geq 1.25$.

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Things to Explore

- Do the roots of a Turnpike polynomial have any special properties? Can we explore empirically?