Topics in Algorithms 2005

Max Cuts

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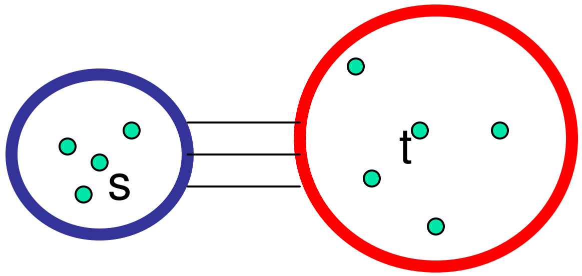
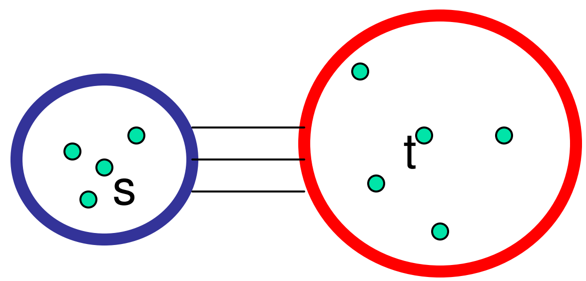
Undirected Unweighted Max Cut

- Cut: Partition of the vertex set
- Max Cut: Number of edges between the two sides of the partition should be maximized



Status

- Max Cut is NP-Complete!! Unlike Min-cut.
- If we can solve this problem we can solve the 3-SAT problem and we can solve any problem which can be solved in non-deterministic polynomial time by a Turing Machine.



NP-Completeness

Show that Max Cut is NP-Complete:

- Reduce from 3-SAT
- Given a 3-SAT formula f find a graph G and a number k such that
  - f is sat implies G has a cut of size k.
  - f is not sat implies G has no cut of size k or larger.

Reduction from 3-SAT

Reduce from 3-SAT:

- Put 2 vertices for each variable x, one for x and one for $\underline{x}$
  - Connect pairs of vertices x,$\underline{x}$ by k edges
- Put one vertex for each clause
  - Connect literals in that clause to it
- The Max Cut will always split complement literal pairs why??
- Ideally, the Max Cut should now equal the number of clauses satisfied plus nk where n is number of variables
- Problems!
  - A clause could have more than 1 satisfying literal
  - Clause vertices could get split as well

Reduction from 3-SAT

Reduce from 3-SAT:

- We need to ensure that each satisfied clause contributes the exact same amount to each cut and each unsatisfied clause contributes a lesser amount. How do we ensure this??

- We need to handle splitting of clause vertices. How?

Trick

- Work with heterogeneous 3-SAT instead
- 3-SAT in which satisfiability requires the assignment to have at least one true and one false in each clause
- Heterogeneous 3-SAT is also NP-Complete. Exercise??

Reduction from hetero-3-SAT

Reduce from hetero-3-SAT with m clauses and n variables:

- Put 2m vertices for each variable x, m for x and m for $\underline{x}$
  - There are 2 vertices associated with each clause
  - Connect all m^2 pairs of vertices x,$\underline{x}$ by k edges each forcing them to come on different sides
- For each clause
  - Connect the three literals for this clause together.
- The Max Cut is knm^2+2#satisfiable clauses. Why??

Approximation Algorithms

- Can we find a near optimal solution in polynomial time?

- E.g., What if we take a random cut? How does it compare to the min-cut?
  - Show that the expected number of edges in the random cut is #edges/2. Exercise!!
  - Thus a random cut is a expected factor ½ approximation

Approximation Algorithms

- How about a greedy algorithm. Switch sides for a vertex if it improves the cut.
  - Will this terminate??
  - What approximation factor does it get??

Beating the ½ factor

- Can you write max-cut as a linear/concave program (so local and global maxima are the same)?

$\text{Max}_{e=(i,j)}$ (1-x_i.x_j)/2

For each vertex x_i, $x_i = -1$ or $x_i = 1$

- Issues
  - Not Linear. Is the objective function concave?
  - Hoe does one relax the –1,1 constraints?
    - Linear relaxation: $x_i >= -1$ and $x_i <= 1$
  - How does one solve a concave problem in polynomial time?
  - How does one round linear solutions to integer solutions?

Concavity

- A twice differentiable function is concave if and only if its Hessian is negative semidefinite, i.e., for vectors v

$$\mathbf{v} \quad \delta^2 f/\delta x_i \delta x_j \quad \mathbf{v} \quad <= 0$$

Maximizing Concave Functions

There are polynomial time algorithms to maximize concave function over any convex set (eg one obtained by intersection of linear inequalities)

- Reference: Solving Convex Programs by Random Walks
  By Bertsimas and Vempala

Beating the ½ factor

$$\text{Max}_{e=(i,j)} \; (1-x_i.x_j)/2$$

For each vertex x_i, $-1 <= x_i <= 1$

- Issues
  - How does one round linear solutions to integer solutions?
  - Think about this? Not sure what is known.

Another Relaxation

$$\text{Max}_{e=(i,j)} \ (1 - x_i . x_j)/2$$

For each vertex x_i, $-1 <= x_i <= 1$

- Relax x_is to be vectors or length 1
- Issues
  - Is the feasible region convex?
  - How does one round?

SemiDefinite Formulation

$$\text{Max}_{e=(i,j)} \quad (1-d_{ij})/2$$

d_{ij} Is pos semidefinite

- The feasible region is convex
- The objective function is linear
- So solvable in polynomial time
- In addition, a nice geometric interpretation
- Due to Goemans and Williamson

SemiDefinite Formulation

$$\mathrm{Max}_{e=(i,j)} \quad (1-d_{ij})/2$$

d_{ij} Is pos semidefinite

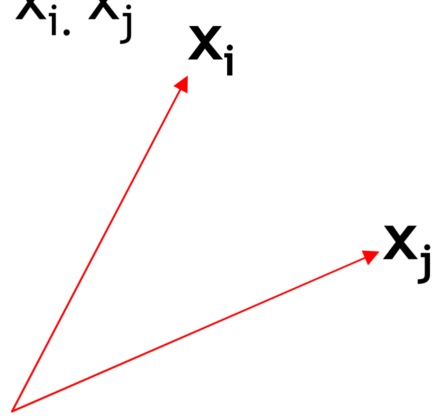
- Rounding
  - Solving the program yields a set of unit length vectors
  - Use a random hyperplane through the origin to split the vectors into 2; this gives a cut
  - How does the size of this cut compare to that of the semidef program optimum??

SemiDefinite Formulation

$\text{Max}_{e=(i,j)} \; (1-d_{ij})/2$

d_{ij} Is pos semidefinite

$d_{ij} = x_i . x_j$



- The probability of the edge e appearing in the cut is $\arccos(d_{ij})/\pi$
- So the approx factor is 2 $\arccos(d_{ij})/\ \pi\ (1-\ d_{ij})$
- How low can $2\theta/\pi\ (1-\cos\theta)$ be?? Exercise??

References

- Bertsimas, Vempala on convex programming

- Goemans-Williamson's paper on Max-Cut

Exercise

- Max-2-SAT
- Given a 2 SAT formula find an assignment which satisfies as many clauses as possible
- Can you give a simple randomized algorithm followed by a SemiDefinite relaxation and then Rounding algorithm?

Thank You