High Dimensional Spaces

Foundations of Data Science Course

Ramesh Hariharan

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What is Volume?

- Volume of a cuboid with sides $l_1, \ldots, l_n$ is $l_1 * l_2 * \cdots * l_n$
- For a general object, integrate:
  - Decompose the object into infinitesimal n-dimensional cuboids
  - Count the number of such cuboids
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Volume of an n-Dimensional Sphere

- $V_n(r) = f_n \times r^n$ for radius r
- $f_1 = 2$
- $f_2 = \pi$
- $f_3 = \frac{4}{3}\pi$
- Does f_n increase or decrease with n?

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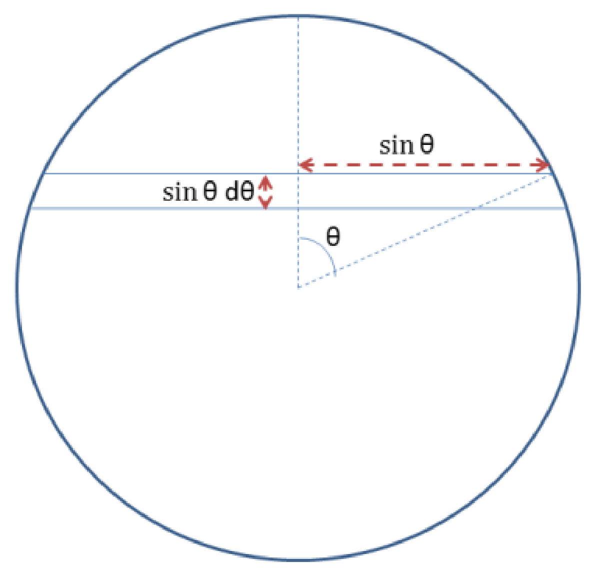
Inductive View of f_n



Inductive Derivation for f_n

- $f_n = 2 \;\; f_{n-1} \;\; \int_0^{\frac{\pi}{2}} \sin^n(\theta)\, d\theta \quad n \geq 2$
- $f_1 = 2$
- $f_n = 2^n \;\; \int_0^{\frac{\pi}{2}} \sin^n(\theta)\, d\theta \;\; \int_0^{\frac{\pi}{2}} \sin^{n-1}(\theta)\, d\theta \;\; \ldots \;\; \int_0^{\frac{\pi}{2}} \sin^1(\theta)\, d\theta$

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Volume of a 1, 2, 3, 4-Dimensional Sphere

- $f_1 = 2$
- $f_2 = 2^2 \quad \int_0^{\frac{\pi}{2}} \sin^2(\theta)\, d\theta \quad \int_0^{\frac{\pi}{2}} \sin^1(\theta)\, d\theta = \pi$
- $f_3 = 2^3 \quad \int_0^{\frac{\pi}{2}} \sin^3(\theta)\, d\theta \quad \int_0^{\frac{\pi}{2}} \sin^2(\theta)\, d\theta \quad \int_0^{\frac{\pi}{2}} \sin^1(\theta)\, d\theta = \frac{4}{3}\pi$
- $f_4 = 2^4 \quad \int_0^{\frac{\pi}{2}} \sin^4(\theta)\, d\theta \quad \int_0^{\frac{\pi}{2}} \sin^3(\theta)\, d\theta \quad \int_0^{\frac{\pi}{2}} \sin^2(\theta)\, d\theta \quad \int_0^{\frac{\pi}{2}} \sin^1(\theta)\, d\theta = \frac{\pi^2}{2}$

Volume of a 1, 2, 3, 4-Dimensional Sphere

- $f_1 = 2$
- $f_2 = 2^2 \int_0^{\frac{\pi}{2}} \sin^2(\theta)\, d\theta \int_0^{\frac{\pi}{2}} \sin^1(\theta)\, d\theta = \pi$
- $f_3 = 2^3 \int_0^{\frac{\pi}{2}} \sin^3(\theta)\, d\theta \int_0^{\frac{\pi}{2}} \sin^2(\theta)\, d\theta \int_0^{\frac{\pi}{2}} \sin^1(\theta)\, d\theta = \frac{4}{3}\pi$
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Sine Power Integrals

- $\int_0^{\frac{\pi}{2}} \sin^n(\theta)d\theta = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2}(\theta)d\theta$
- $\int_0^{\frac{\pi}{2}} \sin^n(\theta)d\theta = \frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{1}{2}\frac{\pi}{2}$, for even n
- $\int_0^{\frac{\pi}{2}} \sin^n(\theta)d\theta = \frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{2}{3}$, for odd n
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- $f_n = \frac{\pi^{(n-1)/2}}{\frac{n}{2}(\frac{n}{2}-1)\cdots\frac{1}{2}}$, for odd n
- $f_n \rightarrow 0$ as $n \rightarrow \infty$!
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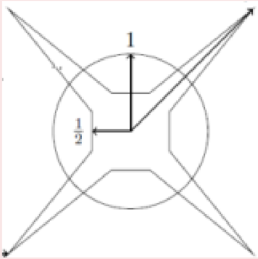
The Unit Sphere vs the Unit Cube

- Corners of a unit cube are distance $\frac{\sqrt{n}}{2}$ from the origin
- Center points of each side are distance $\frac{1}{2}$ from the origin

It looks like this



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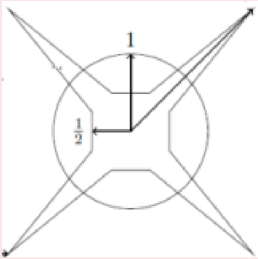

Where is the Volume Concentrated?

- How much of the volume is located outside a band of angle 2α around the equator?
- $\frac{\int_0^{\frac{\pi}{2}-\alpha} \sin^n(\theta)\, d\theta}{\int_0^{\frac{\pi}{2}} \sin^n(\theta)\, d\theta}$
- Denominator: $\int_0^{\frac{\pi}{2}} \sin^n(\theta)\, d\theta \geq \sqrt{\frac{\pi}{2(n+1)}}$
- Numerator: $\int_0^{\frac{\pi}{2}-\alpha} \sin^n(\theta)\, d\theta \leq ?$

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$$= \int_{\sin^2\alpha}^{1} \frac{1}{2\sqrt{y}}(1-y)^{\frac{n-1}{2}}\, dy, \quad y = \cos^2(\theta)$$

$$\leq \frac{1}{2\sin\alpha} \int_{\sin^2\alpha}^{1} e^{-y\frac{n-1}{2}}\, dy$$

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Volume Fraction outside the 2α-angle Equatorial Band

- $$\frac{\int_0^{\frac{\pi}{2}-\alpha} \sin^n(\theta)\, d\theta}{\int_0^{\frac{\pi}{2}} \sin^n(\theta)\, d\theta} \leq \sqrt{\frac{2(n+1)}{\pi}} \frac{1}{(n-1)\sin\alpha} e^{-\frac{n-1}{2}\sin^2\alpha}$$
- For $\alpha \sim \sin(\alpha) = \frac{1}{\sqrt{n}}$, this is $\sim \sqrt{\frac{2}{\pi e}} = .4839$
- More than half the volume is in a $\frac{2}{\sqrt{n}}$ angle band around the equator.
- For $\sin(\alpha) = \frac{a}{\sqrt{n}}$, the above bound is $\sim \sqrt{\frac{2}{\pi}} \frac{1}{a} e^{-\frac{a^2}{2}}$
- Reminiscent of the Normal distribution?
  - $2\int_a^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \leq \sqrt{\frac{2}{\pi}} \frac{1}{a} e^{-\frac{a^2}{2}}$

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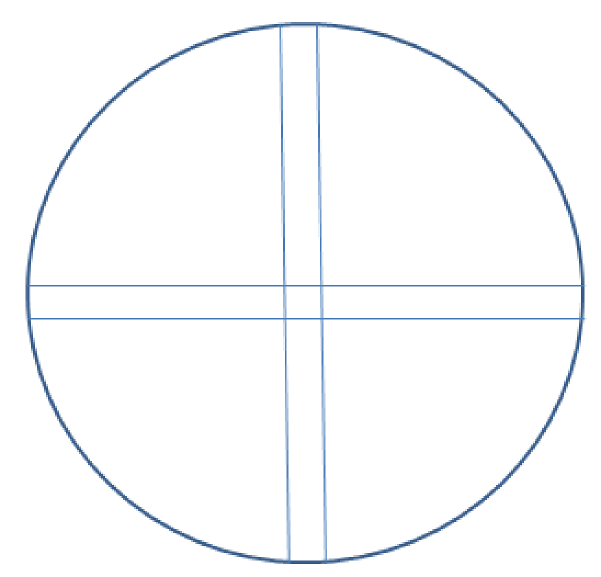
Do 2 Equators sum to more than the whole!



Surface Area $A_n(r)$ of an n-Dimensional Sphere

- $\int_0^r A_n(r)\,dr = V_n(r)$
- $\frac{dV_n(r)}{dr} = A_n(r)$
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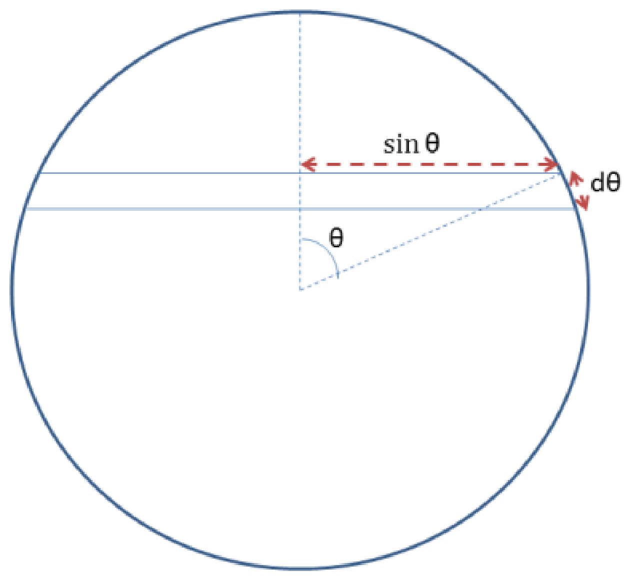
Inductive View of a_n



Dot Product between a Fixed Unit Vector and a Random Unit Vector

- A Spherically Symmetric Random Unit Vector:
  - Probability of lying in any specific patch P on the surface is proportional to the area of P.
- Dot Product is also the length of the projection of the fixed vector on the random vector.
- Dot Product equals $cos(\theta)$, where θ is the angle between the two vectors.
- $E(\cos^2(\theta))$, $Var(\cos^2(\theta))$, and tail bounds on $\cos^2(\theta)$?

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$E(\cos^2(\theta))$

$$\frac{\int_0^{\frac{\pi}{2}} \sin^{n-2}(\theta)\cos^2(\theta)\,d\theta}{\int_0^{\frac{\pi}{2}} \sin^{n-2}(\theta)\,d\theta}$$

$$= \frac{\int_0^{\frac{\pi}{2}} \sin^{n-2}(\theta)\,d\theta - \int_0^{\frac{\pi}{2}} \sin^{n}(\theta)\,d\theta}{\int_0^{\frac{\pi}{2}} \sin^{n-2}(\theta)\,d\theta}$$

$$= 1 - \frac{n-1}{n} = \frac{1}{n}$$

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$$= \frac{\int_0^{\frac{\pi}{2}} \sin^{n-2}(\theta)\,d\theta - 2\int_0^{\frac{\pi}{2}} \sin^{n}(\theta)\,d\theta + \int_0^{\frac{\pi}{2}} \sin^{n+2}(\theta)\,d\theta}{\int_0^{\frac{\pi}{2}} \sin^{n-2}(\theta)\,d\theta} - \frac{1}{n^2}$$

$$= 1 - 2\frac{n-1}{n} + \frac{(n-1)(n+1)}{n(n+2)} - \frac{1}{n^2} = \frac{2(n-1)}{n^2(n+2)} \leq \frac{2}{n^2}$$

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Tail Bounds on $\cos^2(\theta)$

- $Pr(\cos^2(\theta) > \frac{a^2}{n}) = \frac{\int_0^{\cos^{-1}(\frac{a}{\sqrt{n}})} \sin^{n-2}(\theta)\, d\theta}{\int_0^{\frac{\pi}{2}} \sin^{n-2}(\theta)\, d\theta}$
- $\leq \sqrt{\frac{2(n-1)(n-2)}{\pi}} \frac{1}{(n-3)a} e^{-\frac{n-3}{2n}a^2} \sim \sqrt{\frac{2}{\pi}} \frac{1}{a} e^{-\frac{a^2}{2}}$

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Projection Length of Fixed Unit Vector on Random Unit Vector

- With probability $1 - \sqrt{\frac{2}{\pi}}\frac{1}{a}e^{-\frac{a^2}{2}}$, the projected length is between 0 and $\frac{a}{\sqrt{n}}$
- With probability 0.946, the projected length is between 0 and $\frac{2}{\sqrt{n}}$
- Can we drive the projected length to be much more tightly distributed around $\frac{1}{\sqrt{n}}$?

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Project on to many Random Vectors

- Let $X_1, \ldots, X_k$ be the projection lengths on to k independent random unit vectors
- The resulting k-tuple defines a mapping from n-dimensional space to k-dimensional space
- $X = \sqrt{X_1^2 + \cdots + X_k^2}$ is the length of the vector post-mapping
- Consider $X^2 = X_1^2 + \cdots + X_k^2$.

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Sums of Random Variables

- Since $X_1^2, \ldots, X_k^2$ are i.i.d, $E(\frac{X^2}{k}) = E(X_1^2)$ and $Var(\frac{X^2}{k}) = \frac{Var(X_1^2)}{k}$
- I.e., the distribution of $\frac{X^2}{k}$ preserves the mean but is much tighter around the mean.
- $Pr(|\frac{X^2}{k} - E(\frac{X^2}{k})| \geq \alpha) << Pr(|X_1^2 - E(X_1^2)| \geq \alpha)$
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Approximate Length Preservation in k-Dimensional Random Projection

- $E(X^2) = \frac{k}{n}$, by Linearity of Expectation
- $Var(X^2) \leq \frac{2k}{n^2}$, by Linearity of Variance under Independence
- With probability 1−?, X^2 is in $(1-\epsilon)\frac{k}{n} \ldots (1+\epsilon)\frac{k}{n}$
- If ? as small as m^{-3}...
- Union Bound: With probability $1 - m^{-1}$, lengths for m^2 distinct fixed vectors of arbitrary lengths are all simultaneously approximately preserved, modulo scaling by $\sqrt{\frac{n}{k}}$!!

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Asymptotic Tight Concentration for X^2

- By CLT, for $k \to \infty$, the distribution of $X^2 = \sum_0^k X_i^2$ tends to $N(\frac{k}{n}, \leq \frac{2k}{n^2})$
- $Pr(|X^2 - \frac{k}{n}| \geq \epsilon\frac{k}{n})$ should then be $\leq \sqrt{\frac{4}{\epsilon^2 k\pi}} e^{-\frac{\epsilon^2 k}{4}}$
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Tight Concentration and Tail Bound Inequalities

- Markov's Inequality for a non-negative random variable Y

$$Pr(Y > k) \leq E(Y)/k$$

- Chebychev's Inequality

$$Pr(|X^2 - \frac{k}{n}| \geq \epsilon\frac{k}{n}) \leq \frac{Var(X^2)}{(\epsilon\frac{k}{n})^2} \leq \frac{2}{\epsilon^2 k}$$

- Not strong enough to yield negative exponential dependence on k.

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$$E(e^{-tX^2})e^{t(1-\epsilon)\frac{k}{n}} = E(e^{-tX_i^2})^k e^{t(1-\epsilon)\frac{k}{n}}$$

- $E(e^{-tX_i^2}) \leq ?$

Lower Tail Bound for X^2

- Using Markov's inequality on e^{-tX^2}, where $t > 0$ (as in Chernoff Bounds):

$$Pr(X^2 < (1-\epsilon)\frac{k}{n}) = Pr(-tX^2 > -t(1-\epsilon)\frac{k}{n})$$

$$= Pr(e^{-tX^2} > e^{-t(1-\epsilon)\frac{k}{n}}) \leq E(e^{-tX^2})e^{t(1-\epsilon)\frac{k}{n}}$$

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- Using $1 - x \leq e^{-x} \leq 1 - x + \frac{x^2}{2}$, for all $x \geq 0$:

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Completing the Lower Tail Bound for X^2

- $Pr(X^2 < (1-\epsilon)\frac{k}{n}) \leq E(e^{-tX_i^2})^k e^{t(1-\epsilon)\frac{k}{n}}$

$$\leq e^{-\frac{kt}{n}(1-\frac{3t}{2n})+\frac{kt}{n}(1-\epsilon)} \leq e^{-\frac{kt}{n}(\epsilon-\frac{3t}{2n})}$$

- Setting $t = \frac{n\epsilon}{3} > 0$ to minimize the above

$$Pr(X^2 < (1-\epsilon)\frac{k}{n}) \leq e^{-\frac{k\epsilon}{3}(\epsilon-\frac{\epsilon}{2})} \leq e^{-\frac{k\epsilon^2}{6}}$$

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Upper Tail Bound for X^2

- As for the Lower Tail Bound, with $t > 0$:

$$Pr(X^2 > (1+\epsilon)\frac{k}{n}) = Pr(tX^2 > t(1+\epsilon)\frac{k}{n})$$

$$= Pr(e^{tX^2} > e^{t(1+\epsilon)\frac{k}{n}}) \leq E(e^{tX^2})e^{-t(1+\epsilon)\frac{k}{n}}$$

- Since $X^2 = \sum_1^k X_i^2$ and the X_i's are identical and independent:

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The Upper Tail Bound for X^2

- Setting $y = cos^2\theta$.

$$\frac{\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta e^{t\cos^2\theta}\, d\theta}{\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta\, d\theta} \leq \sqrt{\frac{2(n-1)}{\pi}}\frac{1}{2}\int_0^1 \frac{(1-y)^{\frac{n-3}{2}} e^{ty}}{\sqrt{y}}\, dy$$

- Setting $1 - y \leq e^{-y}, \forall y$.

$$\leq \sqrt{\frac{2(n-1)}{\pi}}\frac{1}{2}\int_0^1 \frac{e^{-y(\frac{n-3}{2}-t)}}{\sqrt{y}}\, dy$$

- Setting $\int_0^1 y^{-\frac{1}{2}} e^{-y}\, dy \leq \sqrt{\pi}$

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Completing the Upper Tail Bound for X^2

- $E(e^{tX_i^2})^k e^{-t(1+\epsilon)\frac{k}{n}} \leq (\sqrt{\frac{n-1}{n-3-2t}})^k e^{-t(1+\epsilon)\frac{k}{n}}$
- Using $(1-x)^{-\frac{1}{2}} \leq \sqrt{1+x+2x^2} \leq e^{\frac{x}{2}(1+2x)}$, for $0 \leq x \leq \frac{1}{2}$, and constraining $0 < 2t < \frac{n-3}{2}$, $k << n$

$$(\sqrt{\frac{n-1}{n-3-2t}})^k \leq (\sqrt{\frac{n-1}{n-3}})^k (1 - \frac{2t}{n-3})^{-\frac{k}{2}} \leq e^{O(\frac{k}{n}) + \frac{tk}{n-3}(1+\frac{4t}{n-3})}$$

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- Setting $t = \epsilon\frac{(n-3)^2}{8n}$ and assuming $k << n$, we get:

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Wrapping Up: The Johnson-Lindenstrauß Theorem

- Given m points $a_1, \ldots, a_m$ in n-dimensional space, $m \geq n$, and given ϵ, $0 \leq \epsilon \leq 1$.
- Choose k random unit vectors $r_1, \ldots, r_k$, where $k = \frac{48 \ln m}{\epsilon^2} << n$.
- Define k-dimensional points $b_1, \ldots, b_m$, where $b_i = (a_i \cdot r_1, a_i \cdot r_2, \cdots, a_i \cdot r_k)$.
- Consider any pair a_i, a_j. Then:

$$\frac{|b_i - b_j|}{|a_i - a_j|} = \sqrt{(\frac{a_i - a_j}{|a_i - a_j|} \cdot r_1)^2 + (\frac{a_i - a_j}{|a_i - a_j|} \cdot r_2)^2 + \cdots + (\frac{a_i - a_j}{|a_i - a_j|} \cdot r_n)^2}$$

- Then $\sqrt{(1-\epsilon)}\sqrt{\frac{k}{n}} \leq \frac{|b_i - b_j|}{|a_i - a_j|} \leq \sqrt{(1+\epsilon)}\sqrt{\frac{k}{n}}$ with probability $\frac{3}{m^3}$.
- And this holds for all pairs simultaneously with probability $1 - \frac{3}{2m}$.

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Wrapping Up: The Johnson-Lindenstrauß Theorem

- Given m points $a_1, \ldots, a_m$ in n-dimensional space, $m \geq n$, and given ϵ, $0 \leq \epsilon \leq 1$.
- Choose k random unit vectors $r_1, \ldots, r_k$, where $k = \frac{48 \ln m}{\epsilon^2} << n$.
- Define k-dimensional points $b_1, \ldots, b_m$, where $b_i = (a_i \cdot r_1, a_i \cdot r_2, \cdots, a_i \cdot r_k)$.
- Consider any pair a_i, a_j. Then:

$$\frac{|b_i - b_j|}{|a_i - a_j|} = \sqrt{(\frac{a_i - a_j}{|a_i - a_j|} \cdot r_1)^2 + (\frac{a_i - a_j}{|a_i - a_j|} \cdot r_2)^2 + \cdots + (\frac{a_i - a_j}{|a_i - a_j|} \cdot r_n)^2}$$

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