Topics in Algorithms 2005
The Turnpike Problem

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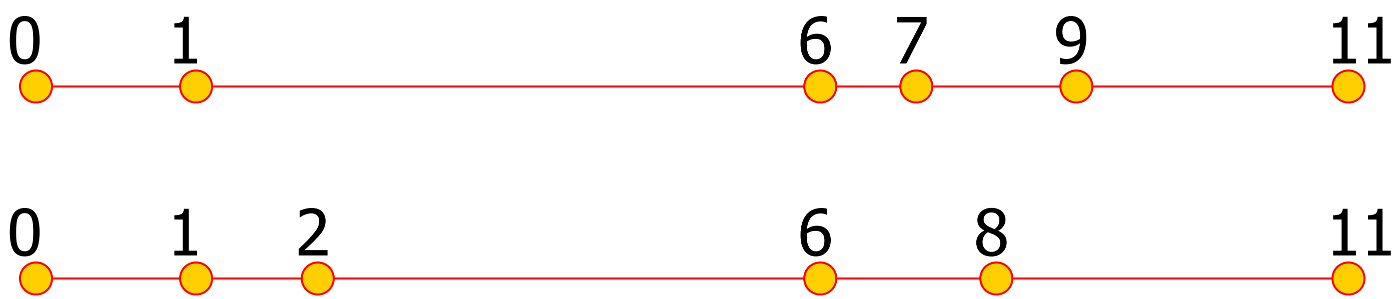
The Problem

- n points on a line
- these define n choose 2 distances
- given points, find distances: easy
- given distances, find points: not so easy

Example

- Distances

  11 10 9 8 7 6 6 5 5 4 3 2 2 1 1



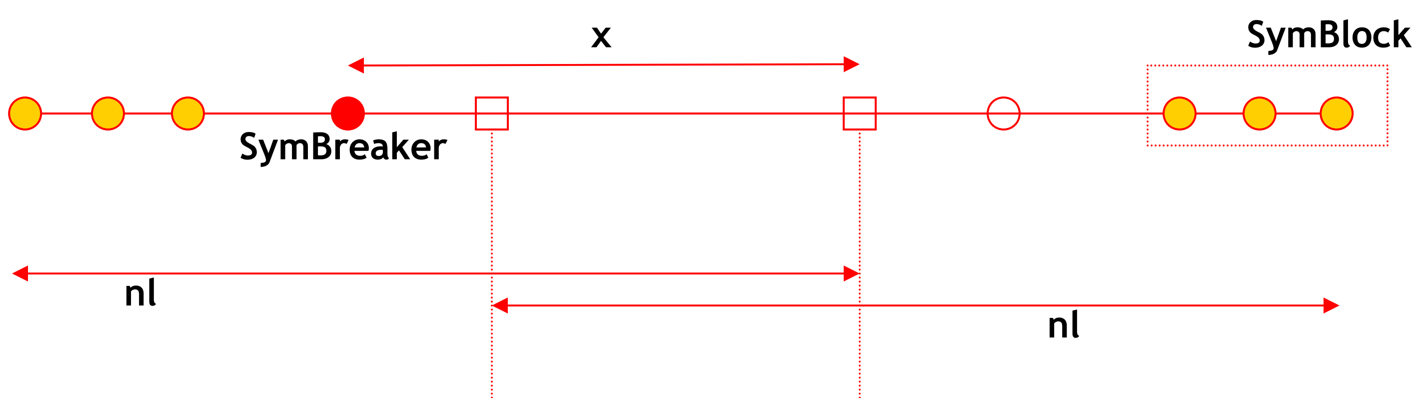
The Outside-In Algorithm

- Largest distance gives leftmost and rightmost points.
- Next largest distance defines the next inside point; should we put it on the left or on the right?
  - 2 copies: both left and right;
  - 1 copy: either is fine by symmetry

- Next largest distance defines the next inside point; should we put it on the left or on the right?
  - 2 copies: both left and right;
  - 1 copy: either is NOT fine as symmetry is already broken
  - How do we choose the right one?
  - Maybe there is no genuine choice, each choice leading to a distinct solution

The Outside-In Algorithm

- Choosing Left or Right
  - Distances from new point to points in the SymBlock: doesn't matter
    Remove these distances from consideration
  - Distance x to the symmetry breaker : matters
    If x does not exist then place on left
    If x exists then: can't say, maybe place on left and then x is realized by a later point along with a point in the symblock



The Outside-In Algorithm

Algorithm

- Two choices at each step, sometimes locally undisambiguable
- Try both choices; take one first and then backtrack if you hit a dead end
- Time: 2^n backtracking alone with each step requiring a dead-end check

dead end check: check if distances from this point to all previous points are available, and then mark these distances as unavailable, requires nlog n time

Total Time: O(2^n nlog n)

Status of the Problem

- Backtracking $2^n n \log n$ is best known provable worst case solution
- Open Problem: Can one do better?

Special Cases

- Suppose the 2n-3 distances which come from endpoints are identified
  - Easy Exercise??

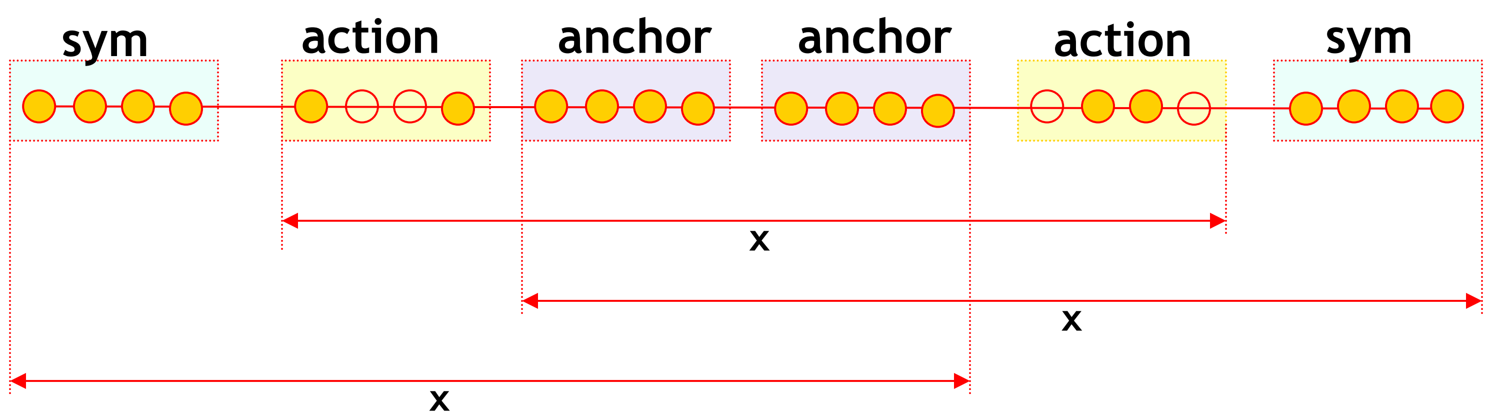
Special Cases

- Suppose all distances are distinct?
  - Not a Hard Exercise??

Hard Example

- Zhang's Example: Backtracking is not of limited depth

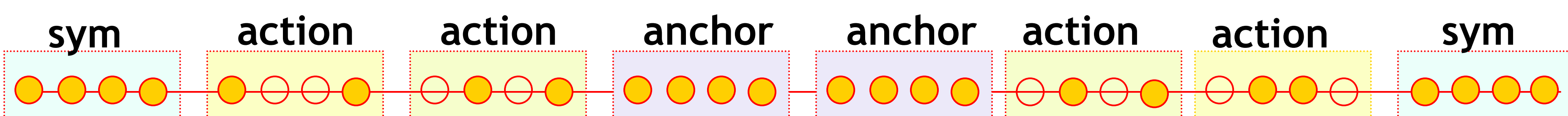
All distances between action blocks are available between the future anchor blocks and the sym blocks



Special Cases

- Zhang's Example: Solvable in polynomial time!!!
  hint: do 2 passes

  first pass: for each locus, identify whether or not it is symmetric
  second pass: do the actual placement



How about these extended Zhang examples
Why doesn't the 2 pass work here?
Open? Could be one way to approach the problem.

A Novel Approach

- Polynomial Representation
  - $d_1 \ldots d_m$ are the distances
  - $p_1 \ldots p_n$ are the point locations
  - $P(x) = \Sigma_k x^{p_k}$
  - Then
  - $P(x)P(1/x) = \Sigma_k x^{p_k} \Sigma_k x^{-p_k}$

$$= \Sigma_{i,j} x^{p_i - p_j}$$

$$= \left[\Sigma_m (x^{d_m} + x^{-d_m}) \right] + n,$$ this is known, call it D(x)

So turnpike comes down to factoring D(x) into factors P(x) and P(1/x) with integer or even 0/1 coefficients.

Turnpike via Polynomial Factorization

- Integer Polynomial Factorization runs in time polynomial in the degree
- Degree of D(x) is the largest distance
- So if the degree of D(x) is small, say poly in n, then we have a polynomial time algorithm for the turpike problem.
- What happens if the degree of D(x) is large? Degree could be exponential in n or even super exponential (I.e., 2^{n^2}) while keeping the problem size polynomial in n.
- So how do we factor high degree polynomials in time sub-exponential in the number of monomials rather than the degree??

Bounding Degrees

- Transform the given set of distances to a set of smaller distances so that the solution set does not change. Denote this transformation by T()

- Key Observation: Suffices to ensure that

$d_i + d_j = d_k$ implies $T(d_i)+T(d_j)=T(d_k)$ for all triples of distances d_i d_j d_k

Proof: Exercise??

- Now one solves the above set of linear equations, numbers in the solution have size at most 6^{(n-1)/2}

Proof: Exercise?? Hint, rewrite all equations in terms of just n-1 distances, I.e., those from the left endpoint; so this system has rank only n-1 and up to 6 terms in each equation; use cramer's rule and hadamard's ineq.

How Many Solutions?

How many ways can an integer polynomial D(x) be factorized over integers into the form P(x)P(1/x)?

- Note D(x) is reversible, i.e., D(x) = deg(D(x)) D(1/x)
- D(x) is uniquely factorizable into irreducible factors over the integers.

  Proof: Exercise??
- Each irreducible factor is either reversible or irreversible.
- If $P_i(x)$ is an irreversible factor, then $P_i(1/x)$ is also an irreversible factor
- Reversible factors must repeat an even number of times (provided there is a solution).
- The total number of distinct solutions is 2^{number of irreversible factors}.
- The number of irreversible factors is O(log n)
- So the number of distinct solutions is just polynomial in n!!!

Number of Irreversible Factors?

- Define a certain measure of a polynomial
- Show the measure of D(x) is small, i.e., bounded by not the degree but the number of terms, or the sum of absolute term coefficients, or sum of squares o these term coefficients.
- Show that the measure is multiplicative, so measure of product is product of measure when you multiply polynomials
- Show that an irreversible polynomial has measure at least 1+x for some fixed value x.
- This implies the bound of O(log n) irreversible factors.

So What

- The number of solutions to a turnpike instance is polynomial in n, which is roughly the input size.
- Such problems belong to the complexity class FewP which is likely to be a strict subset of NP. So Turnpike is unlikley to be NP-Hard
- Does it give a subexponential time algorithm?? Not yet.
- We have a algebraic number theoretic approach which yields a subexp algorithm, assuming a certain number theoretic fact. A overview in later classes.

References

- Skiena, Smith, Lemke: 95, show the bound on the number of solutions, must read paper, available on the web
- Special cases: Distances from endpoints identified
  Pandurangan,Ramesh: on my website
- Unique distances and the alg num th approach
  Mangesh, Naidu,Ramesh: being written up, also in Mangesh's thesis

Thank You